Visual Numerics™

IMSL®

Fortran Subroutines for Mathematical Applications

Math/Library

Special Functions

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Math Library

Special Functions

Version	Revision History	Year	Part Number
1.0	Original Issue	1984	IMSL-SFUN-0001
1.1	Fixed bugs and added significant changes to functionality.	1986	IMSL-SFUN-001.1
2.1	Added routines to enhance functionality.	1991	SFLB-USM-UNBND-EN8901-.21
3.0	No changes were made / reprint only	1994	5111A

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Click here to go to F77/Stat Vol. 1/Library
Click here to go to F90 MP Library
Click here to go to F77/Stat Vol. 2/Library

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Introduction

The IMSL Libraries

The IMSL Libraries consist of two separate, but coordinated Libraries that allow easy user access. These Libraries are organized as follows:

- MATH/LIBRARY general applied mathematics and special functions
- STAT/LIBRARY statistics

The *IMSL MATH/LIBRARY User’s Manual* has two parts: MATH/LIBRARY and MATH/LIBRARY Special Functions.

Most of the routines are available in both single and double precision versions. The same user interface is found on the many hardware versions that span the range from personal computer to supercomputer. Note that some IMSL routines are not distributed for FORTRAN compiler environments that do not support double precision complex data. The names of the IMSL routines that return or accept the type double complex begin with the letter “`Z`” and, occasionally, “`DC`.”

Getting Started

IMSL MATH/LIBRARY Special Functions is a collection of FORTRAN subroutines and functions useful in research and statistical analysis. Each routine is designed and documented to be used in research activities as well as by technical specialists.

To use any of these routines, you must write a program in FORTRAN (or possibly some other language) to call the MATH/LIBRARY Special Functions routine. Each routine conforms to established conventions in programming and documentation. We give first priority in development to efficient algorithms, clear documentation, and accurate results. The uniform design of the routines makes it easy to use more than one routine in a given application. Also, you will find that the design consistency enables you to apply your experience with one MATH/LIBRARY Special Functions routine to all other IMSL routines that you use.

Finding the Right Routine

The organization of IMSL MATH/LIBRARY Special Functions closely parallels that of the National Bureau of Standards' *Handbook of Mathematical Functions*, edited by Abramowitz and Stegun (1964). Corresponding to the NBS Handbook, functions are arranged into separate chapters, such as elementary functions, trigonometric and hyperbolic functions, exponential integrals, gamma function and related functions, and Bessel functions. To locate the right routine for a given problem, you may use either the table of contents located in each chapter introduction, or one of the indexes at the end of this manual. GAMS index uses GAMS classification (Boisvert, R.F., S.E. Howe, D.K. Kahaner, and J.L. Springmann 1990, *Guide to Available Mathematical Software*, National Institute of Standards and Technology NISTIR 90-4237). Use the GAMS index to locate which MATH/LIBRARY Special Functions routines pertain to a particular topic or problem.

Organization of the Documentation

This manual contains a concise description of each routine, with at least one demonstrated example of each routine, including sample input and results. You will find all information pertaining to IMSL MATH/LIBRARY Special Functions in this manual. Moreover, all information pertaining to a particular routine is in one place within a chapter. Each chapter begins with a table of contents that lists the routines included in the chapter. Documentation of the routines consists of the following information.

- IMSL Routine Name
- Purpose: a statement of the purpose of the routine
- Usage: the form for referencing the subprogram with arguments listed. There are two usage forms:
  – `CALL sub(argument-list)` for subroutines
  – `fun(argument-list)` for functions
- Arguments: a description of the arguments in the order of their occurrence. Input arguments usually occur first, followed by input/output arguments, with output arguments described last. For functions, the function symbolic name is described after the argument descriptions.

  Input Argument must be initialized; it is not changed by the routine.

  Input/Output Argument must be initialized; the routine returns output through this argument; cannot be a constant or an expression.

  Input or Output Select appropriate option to define the argument as either input or output. See individual routines for further instructions.

Output No initialization is necessary; cannot be a constant or an expression. The routine returns output through this argument.

- Remarks: details pertaining to code usage and workspace allocation
- Algorithm: a description of the algorithm and references to detailed information. In many cases, other IMSL routines with similar or complementary functions are noted.
- Programming notes: an optional section that contains programming details not covered elsewhere
- Example: at least one application of this routine showing input and required dimension and type statements
- Output: results from the example(s)
- References: periodicals and books with details of algorithm development

Naming Conventions

The names of the routines are mnemonic and unique. Most routines are available in both a single precision and a double precision version, with names of the two versions sharing a common root. The name of the double precision version begins with a "D." The single precision version is generally just the mnemonic root, but sometimes a letter "S" or "A" is used as a prefix. Where possible, we use the letter "C" as a prefix to denote a routine that returns (or accepts) arguments of complex type and the letters "Z" or "DC" for double complex type. For example, the following pairs are names of routines in the two different precisions: ERF/DERF (the root is ERF, for "error function"), ANORDF/DNORDF (the root is NORDF, for "normal distribution function"), and AKER0/DKER0 (the root is KER0, which is the designation of the modified Kelvin function of order 0). The use of the prefix "C" is illustrated by CWPL/ZWPL (the root is WPL, for "Wierstrass P-function, lemniscatic case").

Except when expressly stated otherwise, the names of the variables in the argument lists follow the FORTRAN default type for integer and floating point. In other words, a variable whose name begins with one of the letters "I" through "N" is of type INTEGER, and otherwise is of type REAL or DOUBLE PRECISION, depending on the precision of the routine.

When writing programs accessing IMSL MATH/LIBRARY Special Functions, the user should choose FORTRAN names that do not conflict with names of IMSL subroutines, functions, or named common blocks. The careful user can avoid any conflicts with IMSL names if, in choosing names, the following rules are observed:

- Do not choose a name that appears in the Alphabetical Summary of Routines, at the end of the *User's Manual*.

- Do not choose a name consisting of more than three characters with a numeral in the second or third position.

  For further details, see the section on “Reserved Names” in the Reference Material.

Programming Conventions

In general, the IMSL MATH/LIBRARY Special Functions codes are written so that computations are not affected by underflow, provided the system (hardware or software) places a zero value in the register. In this case, system error messages indicating underflow should be ignored.

IMSL codes also are written to avoid overflow. A program that produces system error messages indicating overflow should be examined for programming errors such as incorrect input data, mismatch of argument types, or improper dimensioning.

In many cases, the documentation for a routine points out common pitfalls that can lead to failure of the algorithm.

Library routines detect error conditions, classify them as to severity, and treat them accordingly. This error-handling capability provides automatic protection for the user without requiring the user to make any specific provisions for the treatment of error conditions. See the section on “User Errors” in the Reference Material for further details.

The routines in IMSL MATH/LIBRARY Special Functions make use of only a few machine constants at run time to initialize various parameters to the particular machine on which they are executing. These machine constants, the most important of which are two machine epsilons and the smallest and largest machine-representable positive numbers, are obtained from three machine-constants routines that have been tailored specifically to the environment in which MATH/LIBRARY Special Functions is being used. Because you may wish to use these routines in your own applications, they are fully discussed in the Reference Material. IMSL MATH/LIBRARY Special Functions does not contain any of the intrinsic functions that are defined to be part of the FORTRAN 77 standard (1978, American National Standard Programming Language FORTRAN, published by American National Standards Institute, New York). Certain local implementations of the FORTRAN compiler may include intrinsic functions in addition to those in the ANSI standard that may also be in MATH/LIBRARY Special Functions. You can check your compiler manual and the table of contents to see if there are any other routines in common.

Error Handling

The routines in IMSL MATH/LIBRARY Special Functions attempt to detect and report errors and invalid input. Errors are classified and are assigned a code number. By default, errors of moderate or worse severity result in messages being automatically printed by the routine. Moreover, errors of worse severity cause program execution to stop. The severity level as well as the general nature of the error is designated by an “error type” with numbers from 0 to 5. An error type 0 is no error; types 1 through 5 are progressively more severe. In most cases, you need not be concerned with our method of handling errors. For those interested, a complete description of the error-handling system is given in the Reference Material, which also describes how you can change the default actions and access the error code numbers.

Work Arrays

A few routines in the IMSL MATH/LIBRARY Special Functions require work arrays. On most systems, the workspace allocation is handled transparently, but on some systems, workspace is obtained from a large array in a `COMMON` block. On these systems, when you have a very large problem, the default workspace may be too small. The routine will print a message telling you the statements to insert in your program in order to provide the needed space (using the common block `WORKSP` for integer or real numbers, or the common block `WKSPCH` for characters). The routine will then automatically halt execution. See “Automatic Workspace Allocation” in the Reference Material for details on common block names and default sizes. For each routine that uses workspace, a second routine is available that allows you to provide the workspace explicitly. For example, the routine `BSJS` (page 103) uses workspace and automatically allocates the required amount, if available. The routine `B2JS` does the same as `BSJS` but has a work array in its argument list, which the user must declare to be of appropriate size. The “Automatic Workspace Allocation” section in the Reference Material contains further details on this subject.

Printing Results

None of the routines in IMSL MATH/LIBRARY Special Functions print results (but error messages may be printed). The output is returned in FORTRAN variables, and you can print these yourself.

The IMSL routine `UMACH` (page 242) retrieves the FORTRAN device unit number for printing. Because this routine obtains device unit numbers, it can be used to redirect the input or output. The section on “Machine-Dependent Constants” in the Reference Material contains a description of the routine `UMACH`.

Chapter 1: Elementary Functions

Routines

Evaluate the argument of a complex number .... CARG 1
Evaluate the cube root of a real number $\sqrt[3]{x}$ .... CBRT 2
Evaluate the cube root of a complex number $\sqrt[3]{z}$ .... CCBRT 3
Evaluate $(e^x - 1)/x$ for real x .... EXPRL 4
Evaluate $(e^z - 1)/z$ for complex z .... CEXPRL 5
Evaluate the complex base 10 logarithm, $\log_{10} z$ CLOG10 6
Evaluate $\ln(x + 1)$ for real x...ALNREL 6
Evaluate $\ln(z + 1)$ for complex z.......................................CLNREL 7

Usage Notes

The “relative” functions `EXPRL` (page 4) and `CEXPRL` (page 5) are useful for accurately computing $e^x - 1$ near $x = 0$. Computing $e^x - 1$ using `EXP(X)` – 1 near $x = 0$ is subject to large cancellation errors.

Similarly, `ALNREL` (page 6) and `CLNREL` (page 7) can be used to accurately compute $\ln(x + 1)$ near $x = 0$. Using the routine `ALOG` to compute $\ln(x + 1)$ near $x = 0$ is subject to large cancellation errors in the computation of 1 + `X`.

CARG/ZARG (Single/Double precision)

Evaluate the argument of a complex number.

Usage

`CARG(Z)`

Arguments

Z — Complex number for which the argument is to be evaluated. (Input)

CARG — Function value.   (Output)
If $z = x + iy$, then arctan(y/x) is returned except when both x and y are zero. In this case, zero is returned.

Algorithm

Arg(z) is the angle θ in the polar representation $z = |z|\, e^{i\theta}$, where

$$i = \sqrt{-1}$$

If $z = x + iy$, then $\theta = \tan^{-1}(y/x)$ except when both x and y are zero. In this case, θ is defined to be zero.

Example

In this example, Arg(1 + i) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       CARG, VALUE
      COMPLEX    Z
      EXTERNAL   CARG, UMACH
C                                  Compute
      Z     = (1.0, 1.0)
      VALUE = CARG(Z)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CARG(', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
CARG( 1.000, 1.000) =  0.785
```

CBRT/DCBRT (Single/Double precision)

Evaluate the cube root.

Usage

`CBRT(X)`

Arguments

X — Argument for which the cube root is desired.   (Input)

CBRT — Function value.   (Output)

Algorithm

The function `CBRT(X)` evaluates $x^{1/3}$. All arguments are legal.

Example

In this example, the cube root of 3.45 is computed and printed.

```
C                                   Declare variables
      INTEGER    NOUT
      REAL       CBRT, VALUE, X
      EXTERNAL   CBRT, UMACH
C                                   Compute
      X     = 3.45
      VALUE = CBRT(X)
C                                   Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' CBRT(', F6.3, ') = ', F6.3)
      END
```

Output

```
CBRT( 3.450) =  1.511
```

CCBRT/ZCBRT (Single/Double precision)

Evaluate the complex cube root.

Usage

`CCBRT(Z)`

Arguments

Z — Complex argument for which the cube root is desired.   (Input)

CCBRT — Complex function value.   (Output)

Comments

The branch cut for the cube root is taken along the negative real axis. The argument of the result, therefore, is greater than $-\pi/3$ and less than or equal to $\pi/3$. The other two roots are obtained by rotating the principal root by $2\pi/3$ and $\pi/3$.

Algorithm

The function `CCBRT(Z)` evaluates $z^{1/3}$. The value $|z|$ must not overflow.

Example

In this example, the cube root of $-3 + 0.0076i$ is computed and printed.

```
C                                   Declare variables
      INTEGER    NOUT
      COMPLEX    CCBRT, VALUE, Z
      EXTERNAL   CCBRT, UMACH
C                                   Compute
```

```
      Z     = (-3.0, 0.0076)
      VALUE = CCBRT(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CCBRT((', F7.4, ',', F7.4, ')) = (',
     &     F6.3, ',' F6.3, ')')
      END
```

Output

```
CCBRT((-3.0000, 0.0076)) = ( 0.722, 1.248)
```

EXPRL/DEXPRL (Single/Double precision)

Evaluate the exponential function factored from first order, (EXP(X) – 1.0)/X.

Usage

EXPRL(X)

Arguments

X — Argument for which the function value is desired. (Input)

EXPRL — Function value. (Output)

Algorithm

The function EXPRL(X) evaluates $(e^x - 1)/x$. It will overflow if e^x overflows.

Example

In this example, EXPRL(0.184) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       EXPRL, VALUE, X
      EXTERNAL   EXPRL, UMACH
C                                 Compute
      X     = 0.184
      VALUE = EXPRL(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' EXPRL(', F6.3, ') = ', F6.3)
      END
```

Output

```
EXPRL( 0.184) =  1.098
```

CEXPRL

Evaluate the complex exponential function factored from first order.

Usage

`CEXPRL(Z)`

Arguments

Z — Complex argument for which the function value is desired. (Input)

CEXPRL — Function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `CEXPRL(Z)` is accurate to less than one-half precision because the complex argument is too close to a nonzero integer multiple of $2\pi i$.

Algorithm

The function `CEXPRL(Z)` evaluates $(e^z - 1)/z$. The argument z must not be so close to a multiple of $2\pi i$ that substantial significance is lost due to cancellation. Also, the result must not overflow and $|\Im z|$ must not be so large that the trigonometric functions are inaccurate.

Example

In this example, `CEXPRL`$(0.0076i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CEXPRL, VALUE, Z
      EXTERNAL   CEXPRL, UMACH
C                                 Compute
      Z     = (0.0, 0.0076)
      VALUE = CEXPRL(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CEXPRL((', F7.4, ',', F7.4, ')) = (',
     &     F6.3, ',' F6.3, ')')
      END
```

Output

```
CEXPRL(( 0.0000, 0.0076)) = ( 1.000, 0.004)
```

CLOG10/ZLOG10 (Single/Double precision)

Evaluate the principal value of the complex common logarithm.

Usage

CLOG10(Z)

Arguments

Z — Complex argument for which the function value is desired. (Input)

CLOG10 — Complex function value. (Output)

Algorithm

The function CLOG10(Z) evaluates $\log_{10}(z)$. The argument must not be zero, and $|z|$ must not overflow.

Example

In this example, the $\log_{10}(0.0076i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CLOG10, VALUE, Z
      EXTERNAL   CLOG10, UMACH
C                                 Compute
      Z     = (0.0, 0.0076)
      VALUE = CLOG10(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CLOG10((', F7.4, ',', F7.4, ')) = (',
     &     F6.3, ',' F6.3, ')')
      END
```

Output

```
CLOG10(( 0.0000, 0.0076)) = (-2.119, 0.682)
```

ALNREL/DLNREL (Single/Double precision)

Evaluate the natural logarithm of one plus the argument.

Usage

ALNREL(X)

Arguments

X — Argument for the function. (Input)

ALNREL — Function value. (Output)

Comments

1. Informational error

Type	Code	
3	2	Result of `ALNREL(X)` is accurate to less than one-half precision because `X` is too near –1.0.

2. `ALNREL` evaluates the natural logarithm of (1 + `X`) accurate in the sense of relative error even when `X` is very small. This routine (as opposed to the intrinsic `ALOG`) should be used to maintain relative accuracy whenever `X` is small and accurately known.

Algorithm

The function `ALNREL(X)` evaluates $\ln(1 + x)$ for $x > -1$. The argument x must be greater than –1.0 to avoid evaluating the logarithm of zero or a negative number. In addition, x must not be so close to –1.0 that considerable significance is lost in evaluating $1 + x$.

Example

In this example, ln(1.189) = `ALNREL`(0.189) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       ALNREL, VALUE, X
      EXTERNAL   ALNREL, UMACH
C                                  Compute
      X     = 0.189
      VALUE = ALNREL(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ALNREL(', F6.3, ') = ', F6.3)
      END
```

Output

```
ALNREL( 0.189) =  0.173
```

CLNREL

Evaluate the principal value of the complex natural logarithm of one plus the argument.

Usage

`CLNREL(Z)`

Arguments

Z — Complex argument for which the complex natural logarithm of 1 + `Z` is desired. (Input)

CLNREL — The complex natural logarithm of (1 + `Z`) accurate in the sense of relative error even when `Z` is small. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `CLNREL(Z)` is accurate to less than one-half precision because `Z` is too near –1.0.

Algorithm

The function `CLNREL(Z)` evaluates $\ln(1 + z)$. The argument z must not be so close to –1 that considerable significance is lost in evaluating $1 + z$. If it is, a recoverable error is issued; however, $z = -1$ is a fatal error because $\ln(1 + z)$ is infinite. Finally, $|z|$ must not overflow.

Let $\rho = |z|$, $z = x + iy$ and $r^2 = |1 + z|^2 = (1 + x)^2 + y^2 = 1 + 2x + \rho^2$. Now, if ρ is small, we may evaluate `CLNREL(Z)` accurately by

$$\begin{aligned}\log(1 + z) &= \log r + i\mathrm{Arg}(z + 1)\\ &= 1/2 \log r^2 + i\mathrm{Arg}(z + 1)\\ &= 1/2\ \mathtt{ALNREL}(2x + \rho^2) + i\mathtt{CARG}(1 + z)\end{aligned}$$

Example

In this example, $\ln(0.0076i)$ = `CLNREL`$(-1 + 0.0076i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CLNREL, VALUE, Z
      EXTERNAL   CLNREL, UMACH
C                                 Compute
      Z     = (-1.0, 0.0076)
      VALUE = CLNREL(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CLNREL((', F6.4, ',', F6.4, ')) = (',
     &     F6.4, ',' F6.4, ')')
      END
```

Output

```
CLNREL((-1.000, .0076)) = (-4.880, 1.571)
```

Chapter 2: Trigonometric and Hyperbolic Functions

Routines

2.1 Trigonometric Functions

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Evaluate cot z for complex z .... CCOT 12
Evaluate sin x for x a real angle in degrees .... SINDG 13
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Evaluate $\cos^{-1} z$ for complex z .... CACOS 16
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2.2 Hyperbolic Functions

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2.3 Inverse Hyperbolic Functions

Evaluate $\sinh^{-1} x$ for real x .... ASINH 21
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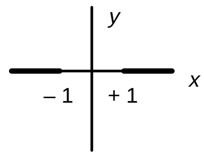
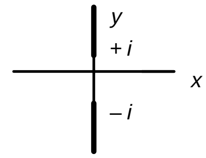
Evaluate $\tanh^{-1} z$ for complex z .... CATANH 25

Usage Notes

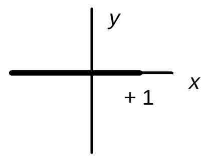
The complex inverse trigonometric hyperbolic functions are single-valued and regular in a slit complex plane. The branch cuts are shown below for $z = x + iy$, i.e., $x = \Re z$ and $y = \Im z$ are the real and imaginary parts of z, respectively.



$\sin^{-1} z$, $\cos^{-1} z$ and $\tanh^{-1}(z)$



$\tan^{-1} z$ and $\sinh^{-1} z$



$\cosh^{-1} z$

Branch Cuts for Inverse Trigonometric and Hyperbolic Functions

CTAN/ZTAN (Single/Double precision)

Evaluate the complex tangent.

Usage

`CTAN(Z)`

Arguments

Z — Complex number representing the angle in radians for which the tangent is desired. (Input)

CTAN — Complex function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `CTAN(Z)` is accurate to less than one-half precision because the real part of `Z` is too near $\pi/2$ or $3\pi/2$ when the imaginary part of `Z` is near zero or because the absolute value of the real part is very large and the absolute value of the imaginary part is small.

Algorithm

Let $z = x + iy$. If $|\cos z|^2$ is very small, that is, if x is very close to $\pi/2$ or $3\pi/2$ and if y is small, then $\tan z$ is nearly singular and a fatal error condition is reported. If $|\cos z|^2$ is somewhat larger but still small, then the result will be less accurate than half precision. When $2x$ is so large that $\sin 2x$ cannot be evaluated to any nonzero precision, the following situation results. If $|y| < 3/2$, then `CTAN` cannot be evaluated accurately to better than one significant figure. If $3/2 \le |y| < -1/2 \ln \varepsilon/2$, then `CTAN` can be evaluated by ignoring the real part of the argument; however, the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision.

Example

In this example, $\tan(1 + i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CTAN, VALUE, Z
      EXTERNAL   CTAN, UMACH
C                                 Compute
      Z     = (1.0, 1.0)
      VALUE = CTAN(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CTAN((', F6.3, ',', F6.3, ')) = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CTAN(( 1.000, 1.000)) = ( 0.272, 1.084)
```

COT/DCOT (Single/Double precision)

Evaluate the cotangent.

Usage

`COT(X)`

Arguments

X — Angle in radians for which the cotangent is desired. (Input)

COT — Function value. (Output)

Comments

1. Informational error
   Type Code

3	2	Result of COT(X) is accurate to less than one-half precision because ABS(X) is too large, or X is nearly a multiple of π.

2. Referencing COT(X) is NOT the same as computing 1.0/TAN(X) because the error conditions are quite different. For example, when X is near π/2, TAN(X) cannot be evaluated accurately and an error message must be issued. However, COT(X) can be evaluated accurately in the sense of absolute error.

Algorithm

The magnitude of x must not be so large that most of the computer word contains the integer part of x. Likewise, x must not be too near an integer multiple of π, although x close to the origin causes no accuracy loss. Finally, x must not be so close to the origin that COT(X) $\approx 1/x$ overflows.

Example

In this example, cot(0.3) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       COT, VALUE, X
      EXTERNAL   COT, UMACH
C                                 Compute
      X     = 0.3
      VALUE = COT(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' COT(', F6.3, ') = ', F6.3)
      END
```

Output

```
COT( 0.300) = 3.233
```

CCOT/ZCOT (Single/Double precision)

Evaluate the complex cotangent.

Usage

CCOT(Z)

Arguments

Z — Complex number representing the angle in radians for which the cotangent is desired. (Input)

CCOT — Complex function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `CCOT(Z)` is accurate to less than one-half precision because the real part of `Z` is too near a multiple of π when the imaginary part of `Z` is near zero, or because the absolute value of the real part is very large and the absolute value of the imaginary part is small

Algorithm

Let $z = x + iy$. If $|\sin z|^2$ is very small, that is, if x is very close to a multiple of π and if $|y|$ is small, then $\cot z$ is nearly singular and a fatal error condition is reported. If $|\sin z|^2$ is somewhat larger but still small, then the result will be less accurate than half precision. When $|2x|$ is so large that $\sin 2x$ cannot be evaluated accurately to even zero precision, the following situation results. If $|y| < 3/2$, then `CCOT` cannot be evaluated accurately to be better than one significant figure. If $3/2 \le |y| < -1/2 \ln \varepsilon/2$, where ε = `AMACH`(4) is the machine precision, then `CCOT` can be evaluated by ignoring the real part of the argument; however, the answer will be less accurate than half precision. Finally, $|z|$ must not be so small that $\cot z \approx 1/z$ overflows.

Example

In this example, $\cot(1 + i)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      COMPLEX    CCOT, VALUE, Z
      EXTERNAL   CCOT, UMACH
C                                  Compute
      Z     = (1.0, 1.0)
      VALUE = CCOT(Z)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CCOT((', F6.3, ',', F6.3, ')) = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CCOT(( 1.000, 1.000)) = ( 0.218,-0.868)
```

SINDG/DSINDG (Single/Double precision)

Evaluate the sine for the argument in degrees.

Usage

`SINDG(X)`

Arguments

X — Argument in degrees for which the sine is desired. (Input)

SINDG — Function value. (Output)

Algorithm

To avoid unduly inaccurate results, the magnitude of *x* must not be so large that the integer part fills more than the computer word. Under no circumstances is the magnitude of *x* allowed to be larger than the largest representable integer because complete loss of accuracy occurs in this case.

Example

In this example, sin 45° is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       SINDG, VALUE, X
      EXTERNAL   SINDG, UMACH
C                                  Compute
      X     = 45.0
      VALUE = SINDG(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' SIN(', F6.3, ' deg) = ', F6.3)
      END
```

Output

```
SIN(45.000 deg) = 0.707
```

COSDG/DCOSDG (Single/Double precision)

Evaluate the cosine for the argument in degrees.

Usage

`COSDG(X)`

Arguments

X — Argument in degrees for which the cosine is desired. (Input)

COSDG — Function value. (Output)

Algorithm

To avoid unduly inaccurate results, the magnitude of *x* must not be so large that the integer part fills more than the computer word. Under no circumstances is the magnitude of *x* allowed to be larger than the largest representable integer because complete loss of accuracy occurs in this case.

Example

In this example, cos 100° computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       COSDG, VALUE, X
      EXTERNAL   COSDG, UMACH
C                                 Compute
      X     = 100.0
      VALUE = COSDG(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' COS(', F6.2, ' deg) = ', F6.3)
      END
```

Output

```
COS(100.00 deg) = -0.174
```

CASIN/ZASIN (Single/Double precision)

Evaluate the complex arc sine.

Usage

`CASIN(ZINP)`

Arguments

ZINP — Complex argument for which the arc sine is desired. (Input)

CASIN — Complex function value in units of radians and the real part in the first or fourth quadrant. (Output)

Algorithm

Almost all arguments are legal. Only when $|z| > b/2$ can an overflow occur. Here, b = `AMACH`(2) is the largest floating point number. This error is not detected by `CASIN`.

See Pennisi (1963, page 126) for reference.

Example

In this example, $\sin^{-1}(1 - i)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      COMPLEX    CASIN, VALUE, Z
      EXTERNAL   CASIN, UMACH
C                                  Compute
      Z     = (1.0, -1.0)
      VALUE = CASIN(Z)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CASIN((', F6.3, ',', F6.3, ')) = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CASIN(( 1.000,-1.000)) = ( 0.666,-1.061)
```

CACOS/ZACOS (Single/Double precision)

Evaluate the complex arc cosine.

Usage

`CACOS(Z)`

Arguments

Z — Complex argument for which the arc cosine is desired. (Input)

CACOS — Complex function value in units of radians with the real part in the first or second quadrant. (Output)

Algorithm

Almost all arguments are legal. Only when $|z| > b/2$ can an overflow occur. Here, b = `AMACH`(2) is the largest floating point number. This error is not detected by `CACOS`.

Example

In this example, $\cos^{-1}(1 - i)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      COMPLEX    CACOS, VALUE, Z
      EXTERNAL   CACOS, UMACH
C                                  Compute
      Z     = (1.0, -1.0)
      VALUE = CACOS(Z)
C                                  Print the results
```

```
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CACOS((', F6.3, ',', F6.3, ')) = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CACOS(( 1.000,-1.000)) = ( 0.905, 1.061)
```

CATAN/ZATAN (Single/Double precision)

Evaluate the complex arc tangent.

Usage

`CATAN(Z)`

Arguments

Z — Complex argument for which the arc tangent is desired. (Input)

CATAN — Complex function value in units of radians with the real part in the first or fourth quadrant. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `CATAN(Z)` is accurate to less than one-half precision because $|Z^2|$ is too close to –1.0.

Algorithm

The argument z must not be exactly $\pm i$, because $\tan^{-1} z$ is undefined there. In addition, z must not be so close to $\pm i$ that substantial significance is lost.

Example

In this example, $\tan^{-1}(0.01 - 0.01i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CATAN, VALUE, Z
      EXTERNAL   CATAN, UMACH
C                                 Compute
      Z     = (0.01, 0.01)
      VALUE = CATAN(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CATAN((', F6.3, ',', F6.3, ')) = (',
     &     F6.3, ',', F6.3, ')')
```

```
      END
```

Output

```
CATAN(( 0.010, 0.010)) = ( 0.010, 0.010)
```

CATAN2/ZATAN2 (Single/Double precision)

Evaluate the complex arc tangent of a ratio.

Usage

CATAN2(CSN, CCS)

Arguments

CSN — Complex numerator of the ratio for which the arc tangent is desired. (Input)

CCS — Complex denominator of the ratio. (Input)

CATAN2 — Complex function value in units of radians with the real part between $-\pi$ and π. (Output)

Comments

The result is returned in the correct quadrant (modulo 2π).

Algorithm

Let z_1 = CSN and z_2 = CCS. The ratio $z = z_1/z_2$ must not be $\pm i$ because $\tan^{-1}(\pm i)$ is undefined. Likewise, z_1 and z_2 should not both be zero. Finally, z must not be so close to $\pm i$ that substantial accuracy loss occurs.

Example

In this example,

$$\tan^{-1}\frac{(1/2)+(i/2)}{2+i}$$

is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CATAN2, VALUE, X, Y
      EXTERNAL   CATAN2, UMACH
C                                 Compute
      X     = (2.0, 1.0)
      Y     = (0.5, 0.5)
      VALUE = CATAN2(Y, X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Y, X, VALUE
```

```
99999 FORMAT (' CATAN2((', F6.3, ',', F6.3, '), (', F6.3, ',', F6.3,
     &       ')) = (', F6.3, ',', F6.3, ')')
      END
```

Output

```
CATAN2(( 0.500, 0.500), ( 2.000, 1.000)) = ( 0.294, 0.092)
```

CSINH/ZSINH (Single/Double precision)

Evaluate the complex hyperbolic sine.

Usage

`CSINH(Z)`

Arguments

Z — Complex number representing the angle in radians for which the complex hyperbolic sine is desired. (Input)

CSINH — Complex function value. (Output)

Algorithm

The argument *z* must satisfy

$$\left|\Im z\right| \le 1/\sqrt{\varepsilon}$$

where ε = `AMACH`(4) is the machine precision and $\Im z$ is the imaginary part of *z*.

Example

In this example, sinh(5 – *i*) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      COMPLEX    CSINH, VALUE, Z
      EXTERNAL   CSINH, UMACH
C                                  Compute
      Z     = (5.0, -1.0)
      VALUE = CSINH(Z)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CSINH((', F6.3, ',', F6.3, ')) = (',
     &       F7.3, ',', F7.3, ')')
      END
```

Output

```
CSINH(( 5.000,-1.000)) = ( 40.092,-62.446)
```

CCOSH/ZCOSH (Single/Double precision)

Evaluate the complex hyperbolic cosine.

Usage

CCOSH(Z)

Arguments

Z — Complex number representing the angle in radians for which the hyperbolic cosine is desired. (Input)

CCOSH — Complex function value. (Output)

Algorithm

Let ε = AMACH(4) be the machine precision. If $|\Im z|$ is larger than

$$1/\sqrt{\varepsilon}$$

then the result will be less than half precision, and a recoverable error condition is reported. If $|\Im z|$ is larger than $1/\varepsilon$, the result has no precision and a fatal error is reported. Finally, if $|\Re z|$ is too large, the result overflows and a fatal error results. Here, $\Re z$ and $\Im z$ represent the real and imaginary parts of z, respectively.

Example

In this example, $\cosh(-2 + 2i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CCOSH, VALUE, Z
      EXTERNAL   CCOSH, UMACH
C                                 Compute
      Z     = (-2.0, 2.0)
      VALUE = CCOSH(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CCOSH((', F6.3, ',', F6.3, ')) = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CCOSH((-2.000, 2.000)) = (-1.566,-3.298)
```

CTANH/ZTANH (Single/Double precision)

Evaluate the complex hyperbolic tangent.

Usage

`CTANH(Z)`

Arguments

Z — Complex number representing the angle in radians for which the hyperbolic tangent is desired. (Input)

CTANH — Complex function value. (Output)

Algorithm

Let $z = x + iy$. If $|\cosh z|^2$ is very small, that is, if y mod 2π is very close to $\pi/2$ or $3\pi/2$ and if x is small, then $\tanh z$ is nearly singular; a fatal error condition is reported. If $|\cosh z|^2$ is somewhat larger but still small, then the result will be less accurate than half precision. When $2y$ ($z = x + iy$) is so large that $\sin 2y$ cannot be evaluated accurately to even zero precision, the following situation results. If $|x| < 3/2$, then `CTANH` cannot be evaluated accurately to better than one significant figure. If $3/2 \le |y| < -1/2 \ln(\varepsilon/2)$, then `CTANH` can be evaluated by ignoring the imaginary part of the argument; however, the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision.

Example

In this example, $\tanh(1 + i)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      COMPLEX    CTANH, VALUE, Z
      EXTERNAL   CTANH, UMACH
C                                  Compute
      Z     = (1.0, 1.0)
      VALUE = CTANH(Z)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CTANH((', F6.3, ',', F6.3, ')) = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CTANH(( 1.000, 1.000)) = ( 1.084, 0.272)
```

ASINH/DASINH (Single/Double precision)

Evaluate the arc hyperbolic sine.

Usage

`ASINH(X)`

Arguments

X — Argument for which the arc hyperbolic sine is desired. (Input)

ASINH — Function value. (Output)

Algorithm

The function `ASINH(X)` computes the inverse hyperbolic sine of x, $\sinh^{-1} x$.

Example

In this example, $\sinh^{-1}(2.0)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ASINH, VALUE, X
      EXTERNAL   ASINH, UMACH
C                                 Compute
      X     = 2.0
      VALUE = ASINH(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ASINH(', F6.3, ') = ', F6.3)
      END
```

Output

```
ASINH( 2.000) = 1.444
```

CASINH/ZASINH (Single/Double precision)

Evaluate the complex arc hyperbolic sine.

Usage

`CASINH(Z)`

Arguments

Z — Complex argument for which the arc hyperbolic sine is desired. (Input)

CASINH — Complex function value. (Output)

Algorithm

Almost all arguments are legal. Only when $|z| > b/2$ can an overflow occur, where b = `AMACH`(2) is the largest floating point number. This error is not detected by `CASINH`.

Example

In this example, $\sinh^{-1}(-1 + i)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      COMPLEX    CASINH, VALUE, Z
      EXTERNAL   CASINH, UMACH
C                                  Compute
      Z     = (-1.0, 1.0)
      VALUE = CASINH(Z)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CASINH((', F6.3, ',', F6.3, ')) = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CASINH((-1.000, 1.000)) = (-1.061, 0.666)
```

ACOSH/DACOSH (Single/Double precision)

Evaluate the arc hyperbolic cosine.

Usage

`ACOSH(X)`

Arguments

X — Argument for which the arc hyperbolic cosine is desired. (Input)

ACOSH — Function value. (Output)

Comments

The result of `ACOSH`(X) is returned on the positive branch. Recall that, like `SQRT`(X), `ACOSH`(X) has multiple values.

Algorithm

The function `ACOSH`(X) computes the inverse hyperbolic cosine of x, $\cosh^{-1} x$.

Example

In this example, $\cosh^{-1}(1.4)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       ACOSH, VALUE, X
      EXTERNAL   ACOSH, UMACH
C                                  Compute
      X     = 1.4
      VALUE = ACOSH(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
```

```
WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ACOSH(', F6.3, ') = ', F6.3)
      END
```

Output

```
ACOSH( 1.400) = 0.867
```

CACOSH/ZACOSH (Single/Double precision)

Evaluate the complex arc hyperbolic cosine.

Usage

`CACOSH(Z)`

Arguments

Z — Complex argument for which the arc hyperbolic cosine is desired. (Input)

CACOSH — Complex function value. (Output)

Algorithm

Almost all arguments are legal. Only when $|z| > b/2$ can an overflow occur, where b = `AMACH`(2) is the largest floating point number. This error is not detected by `CACOSH`.

Example

In this example, $\cosh^{-1}(1 - i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CACOSH, VALUE, Z
      EXTERNAL   CACOSH, UMACH
C                                 Compute
      Z     = (1.0, -1.0)
      VALUE = CACOSH(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CACOSH((', F6.3, ',', F6.3, ')) = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CACOSH(( 1.000,-1.000)) = (-1.061, 0.905)
```

ATANH/DATANH (Single/Double precision)

Evaluate the arc hyperbolic tangent.

Usage

`ATANH(X)`

Arguments

X — Argument for which the arc hyperbolic tangent is desired. (Input)

ATANH — Function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `ATANH(X)` is accurate to less than one-half precision because the absolute value of the argument is too close to 1.0.

Algorithm

`ATANH(X)` computes the inverse hyperbolic tangent of x, $\tanh^{-1} x$. The argument x must satisfy

$$|x| < 1 - \sqrt{\varepsilon}$$

where ε = `AMACH(4)` is the machine precision. Note that $|x|$ must not be so close to one that the result is less accurate than half precision.

Example

In this example, $\tanh^{-1}(-1/4)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ATANH, VALUE, X
      EXTERNAL   ATANH, UMACH
C                                 Compute
      X     = -0.25
      VALUE = ATANH(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ATANH(', F6.3, ') = ', F6.3)
      END
```

Output

```
ATANH(-0.250) = -0.255
```

CATANH/ZATANH (Single/Double precision)

Evaluate the complex arc hyperbolic tangent.

Usage

`CATANH(Z)`

Arguments

Z — Complex argument for which the arc hyperbolic tangent is desired. (Input)

CATANH — Complex function value. (Output)

Algorithm

The argument must not be exactly ±1 because $\tanh^{-1} z$ is undefined there. In addition, z must not be so close to ±1 that substantial significance is lost.

Example

In this example, $\tanh^{-1}(1/2 + i/2)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CATANH, VALUE, Z
      EXTERNAL   CATANH, UMACH
C                                 Compute
      Z     = (0.5, 0.5)
      VALUE = CATANH(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CATANH((', F6.3, ',', F6.3, ')) = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CATANH(( 0.500, 0.500)) = ( 0.402, 0.554)
```

Chapter 3: Exponential Integrals and Related Functions

Routines

Evaluate the exponential integral, Ei(x)..EI 28
Evaluate the exponential integral, $E_1(x)$.......................................E1 29
Evaluate the scaled exponential integrals, integer order, $E_n(x)$..ENE 30
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Usage Notes

The notation used in this chapter follows that of Abramowitz and Stegun (1964).

The following is a plot of the exponential integral functions that can be computed by the routines described in this chapter.

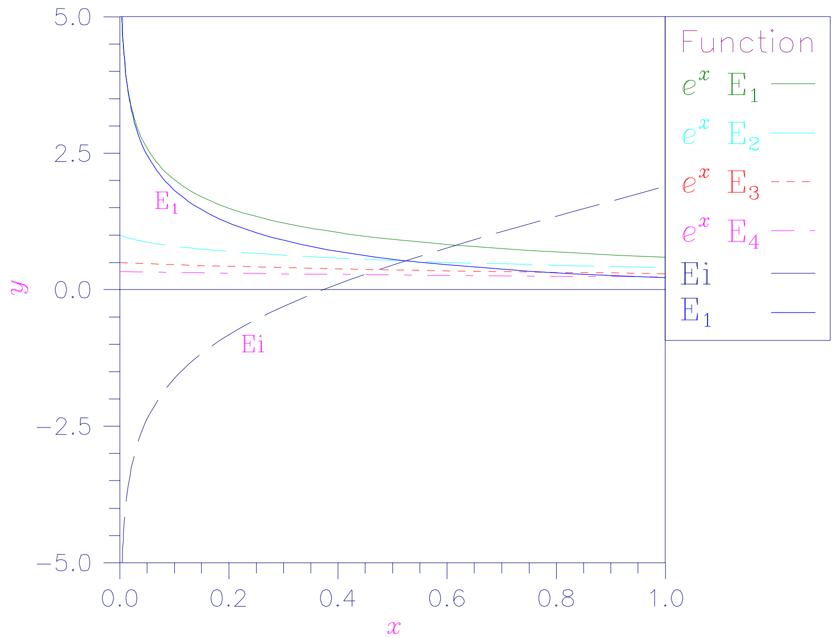

Figure 3-1 Plot of $e^x E(x)$, $E_1(x)$ and $Ei(x)$

EI/DEI (Single/Double precision)

Evaluate the exponential integral for arguments greater than zero and the Cauchy principal value for arguments less than zero.

Usage

`EI(X)`

Arguments

X — Argument for which the function value is desired. (Input)

EI — Function value. (Output)

Comments

If principal values are used everywhere, then for all `X`, `EI(X)` = −`E1`(−`X`) and `E1(X)` = −`EI`(−`X`)

Algorithm

The exponential integral, Ei(x), is defined to be

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} e^{-t} / t \quad dt \quad \text{for } x \neq 0$$

The argument *x* must be large enough to insure that the asymptotic formula e^x/x does not underflow, and *x* must not be so large that e^x overflows.

Example

In this example, Ei(1.15) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       EI, VALUE, X
      EXTERNAL   EI, UMACH
C                                 Compute
      X     = 1.15
      VALUE = EI(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' EI(', F6.3, ') = ', F6.3)
      END
```

Output

```
EI( 1.150) =  2.304
```

E1/DE1 (Single/Double precision)

Evaluate the exponential integral for arguments greater than zero and the Cauchy principal value of the integral for arguments less than zero.

Usage

`E1(X)`

Arguments

X — Argument for which the integral is to be evaluated. (Input)

E1 — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too large.

Algorithm

The alternate definition of the exponential integral, $E_1(x)$, is

$$E_1(x) = \int_x^\infty e^{-t} / t \; dt \quad \text{for } x \neq 0$$

The path of integration must exclude the origin and not cross the negative real axis.

The argument x must be large enough that e^{-x} does not overflow, and x must be small enough to insure that e^{-x}/x does not underflow.

Example

In this example, $E_1(1.3)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       E1, VALUE, X
      EXTERNAL   E1, UMACH
C                                  Compute
      X     = 1.3
      VALUE = E1(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' E1(', F6.3, ') = ', F6.3)
      END
```

Output

```
E1( 1.300) =  0.135
```

ENE/DENE (Single/Double precision)

Evaluate the exponential integral of integer order for arguments greater than zero scaled by `EXP(X)`.

Usage

`CALL ENE (X, N, F)`

Arguments

X — Argument for which the integral is to be evaluated. (Input)
It must be greater than zero.

N — Integer specifying the maximum order for which the exponential integral is to be calculated. (Input)

F — Vector of length `N` containing the computed exponential integrals scaled by `EXP(X)`. (Output)

Algorithm

The scaled exponential integral of order n, $E_n(x)$, is defined to be

$$E_n(x) = e^x \int_1^\infty e^{-xt} t^{-n}\, dt \quad \text{for } x > 0$$

The argument x must satisfy $x > 0$. The integer n must also be greater than zero. This code is based on a code due to Gautschi (1974).

Example

In this example, $E_n(10)$ for $n = 1, ..., n$ is computed and printed.

```
C                                 Declare variables
      INTEGER    N
      PARAMETER  (N=10)
C
      INTEGER    K, NOUT
      REAL       F(N), X
      EXTERNAL   ENE, UMACH
C                                 Compute
      X = 10.0
      CALL ENE (X, N, F)
C                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K, X, F(K)
   10 CONTINUE
99999 FORMAT (' E sub ', I2, ' (', F6.3, ') = ', F6.3)
      END
```

Output

```
E sub  1 (10.000) =  0.092
E sub  2 (10.000) =  0.084
E sub  3 (10.000) =  0.078
E sub  4 (10.000) =  0.073
E sub  5 (10.000) =  0.068
E sub  6 (10.000) =  0.064
E sub  7 (10.000) =  0.060
E sub  8 (10.000) =  0.057
E sub  9 (10.000) =  0.054
E sub 10 (10.000) =  0.051
```

ALI/DLI (Single/Double precision)

Evaluate the logarithmic integral.

Usage

`ALI(X)`

Arguments

X — Argument for which the logarithmic integral is desired. (Input)
It must be greater than zero and not equal to one.

ALI — Function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `ALI(X)` is accurate to less than one-half precision because `X` is too close to 1.0.

Algorithm

The logarithmic integral, li(x), is defined to be

$$\mathrm{li}(x) = -⨍_0^x \frac{dt}{\ln t} \qquad \text{for } x > 0 \text{ and } x \neq 1$$

The argument x must be greater than zero and not equal to one. To avoid an undue loss of accuracy, x must be different from one at least by the square root of the machine precision.

The function li(x) approximates the function $\pi(x)$, the number of primes less than or equal to x. Assuming the Riemann hypothesis (all non-real zeros of $\zeta(z)$ are on the line $\Re z = 1/2$), then

$$\mathrm{li}(x) - \pi(x) = O(\sqrt{x}\ln x)$$

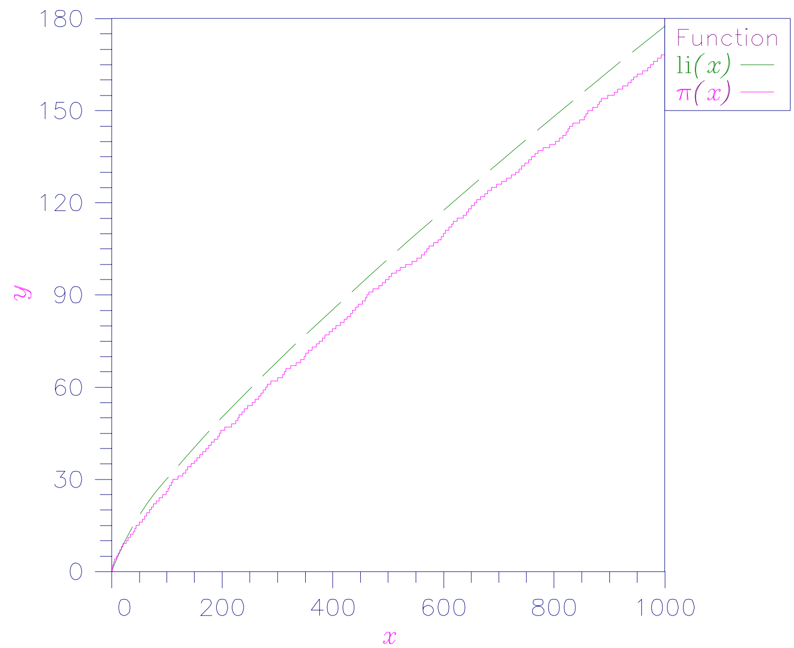

Figure 3-2 Plot of li(x) and $\pi(x)$

Example

In this example, li(2.3) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       ALI, VALUE, X
      EXTERNAL   ALI, UMACH
C                                  Compute
      X     = 2.3
      VALUE = ALI(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ALI(', F6.3, ') = ', F6.3)
      END
```

Output

```
ALI( 2.300) =  1.439
```

SI/DSI (Single/Double precision)

Evaluate the sine integral.

Usage

SI(X)

Arguments

X — Argument for which the function value is desired. (Input)

SI — Function value. (Output)

Algorithm

The sine integral, Si(x), is defined to be

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$$

If

$$|x| > 1/\sqrt{\varepsilon}$$

the answer is less accurate than half precision, while for $|x| > 1/\varepsilon$, the answer has no precision. Here, ε = AMACH(4) is the machine precision.

Example

In this example, Si(1.25) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       SI, VALUE, X
```

```
      EXTERNAL   SI, UMACH
C                                  Compute
      X     = 1.25
      VALUE = SI(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' SI(', F6.3, ') = ', F6.3)
      END
```

Output

```
SI( 1.250) =  1.146
```

CI/DCI (Single/Double precision)

Evaluate the cosine integral.

Usage

`CI(X)`

Arguments

X — Argument for which the function value is desired. (Input)
It must be greater than zero.

CI — Function value. (Output)

Algorithm

The cosine integral, Ci(x), is defined to be

$$\mathrm{Ci}(x) = \gamma + \ln x + \int_0^x \frac{1-\cos t}{t}\,dt$$

where $\gamma \approx 0.57721566$ is Euler's constant.

The argument x must be larger than zero. If

$$x > 1/\sqrt{\varepsilon}$$

then the result will be less accurate than half precision. If $x > 1/\varepsilon$, the result will have no precision. Here, ε = `AMACH`(4) is the machine precision.

Example

In this example, Ci(1.5) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       CI, VALUE, X
      EXTERNAL   CI, UMACH
C                                  Compute
      X     = 1.5
```

```
      VALUE = CI(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' CI(', F6.3, ') = ', F6.3)
      END
```

Output

```
CI( 1.500) =  0.470
```

CIN/DCIN (Single/Double precision)

Evaluate a function closely related to the cosine integral.

Usage

`CIN(X)`

Arguments

X — Argument for which the function value is desired. (Input)

CIN — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too small.

Algorithm

The alternate definition of the cosine integral, Cin(x), is

$$\mathrm{Cin}(x) = \int_0^x \frac{1 - \cos t}{t}\,dt$$

For

$$0 < |x| < \sqrt{s}$$

where s = `AMACH`(1) is the smallest representable positive number, the result underflows. For

$$|x| > 1/\sqrt{\varepsilon}$$

the answer is less accurate than half precision, while for $|x| > 1/\varepsilon$, the answer has no precision. Here, ε = `AMACH`(4) is the machine precision.

Example

In this example, Cin(2π) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       CIN, CONST, VALUE, X
      EXTERNAL   CIN, CONST, UMACH
C                                 Compute
      X     = 2.0*CONST('pi')
      VALUE = CIN(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' CIN(', F6.3, ') = ', F6.3)
      END
```

Output

```
CIN( 6.283) =  2.438
```

SHI/DSHI (Single/Double precision)

Evaluate the hyperbolic sine integral.

Usage

`SHI(X)`

Arguments

X — Argument for which the function value is desired. (Input)

SHI— function value. (Output)
`SHI` equals

$$\int_0^x \sinh(t) / t \, dt$$

Algorithm

The hyperbolic sine integral, Shi(x), is defined to be

$$\mathrm{Shi}(x) = \int_0^x \frac{\sinh t}{t} dt$$

The argument x must be large enough that e^{-x}/x does not underflow, and x must be small enough that e^x does not overflow.

Example

In this example, Shi(3.5) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       SHI, VALUE, X
      EXTERNAL   SHI, UMACH
C                                 Compute
```

```
      X     = 3.5
      VALUE = SHI(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' SHI(', F6.3, ') = ', F6.3)
      END
```

Output

```
SHI( 3.500) =  6.966
```

CHI/DCHI (Single/Double precision)

Evaluate the hyperbolic cosine integral.

Usage

`CHI(X)`

Arguments

X — Argument for which the function value is desired. (Input)

CHI — Function value. (Output)

Comments

When `X` is negative, the principal value is used.

Algorithm

The hyperbolic cosine integral, Chi(x), is defined to be

$$\mathrm{Chi}(x) = \gamma + \ln x + \int_0^x \frac{\cosh t - 1}{t}\, dt \qquad \text{for } x > 0$$

where $\gamma \approx 0.57721566$ is Euler's constant.

The argument x must be large enough that e^{-x}/x does not underflow, and x must be small enough that e^x does not overflow.

Example

In this example, Chi(2.5) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       CHI, VALUE, X
      EXTERNAL   CHI, UMACH
C                                  Compute
      X     = 2.5
      VALUE = CHI(X)
C                                  Print the results
```

```
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' CHI(', F6.3, ') = ', F6.3)
      END
```

Output

```
CHI( 2.500) =  3.524
```

CINH/DCINH (Single/Double precision)

Evaluate a function closely related to the hyperbolic cosine integral.

Usage

CINH(X)

Arguments

X — Argument for which the function value is desired. (Input)

CINH — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because X is too small.

Algorithm

The alternate definition of the hyperbolic cosine integral, Cinh(x), is

$$\mathrm{Cinh}(x) = \int_0^x \frac{\cosh t - 1}{t}\,dt$$

For

$$0 < |x| < 2\sqrt{s}$$

where s = AMACH(1) is the smallest representable positive number, the result underflows. The argument x must be large enough that e^{-x}/x does not underflow, and x must be small enough that e^x does not overflow.

Example

In this example, Cinh(2.5) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       CINH, VALUE, X
      EXTERNAL   CINH, UMACH
C                                 Compute
```

```
      X     = 2.5
      VALUE = CINH(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' CINH(', F6.3, ') = ', F6.3)
      END
```

Output

```
CINH( 2.500) =  2.031
```

Chapter 4: Gamma Function and Related Functions

Routines

4.1 Factorial Function

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Evaluate the binomial coefficient, $\binom{n}{m}$ .... BINOM 43

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4.4. Psi Function

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4.6. Beta Function

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Usage Notes

The notation used in this chapter follows that of Abramowitz and Stegun (1964).

The following is a table of the functions defined in this chapter:

FAC	$n! = \Gamma(n + 1)$
BINOM	$n!/m!(n - m)!,\ 0 \le m \le n$
GAMMA	$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt,\ x \ne 0, -1, -2, \dots$
CGAMMA	$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,\ x \ne 0, -1, -2, \dots$
GAMR	$1/\Gamma(x)$
CGAMR	$1/\Gamma(z)$
ALNGAM	$\ln \lvert\Gamma(x)\rvert,\ x \ne 0, -1, -2, \dots$
CLNGAM	$\ln \Gamma(z),\ x \ne 0, -1, -2, \dots$
ALGAMS	$\ln \lvert\Gamma(x)\rvert$ and sign $\Gamma(x),\ x \ne 0, -1, -2, \dots$
GAMI	$\gamma(a, x) = \int_0^x t^{a-1} e^{-t}\, dt, a > 0,\ x \ge 0$
GAMIC	$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t}\, dt,\ x > 0$
GAMIT	$\gamma^*(a, x) = (x^{-a}/\Gamma(a))\gamma(a, x),\ x \ge 0$
PSI	$\psi(x) = \Gamma'(x)/\Gamma(x),\ x \ne 0, -1, -2, \dots$
CPSI	$\psi(z) = \Gamma'(z)/\Gamma(z),\ z \ne 0, -1, -2, \dots$
POCH	$(a)_x = \Gamma(a + x)/\Gamma(a)$, if $a + x = 0, -1, -2, \dots$ then a must $= 0, -1, -2, \dots$
POCH1	$((a)_x - 1)/x$, if $a + x = 0, -1, -2, \dots$ then a must $= 0, -1, -2, \dots$
BETA	$\beta(x_1, x_2) = \Gamma(x_1)\Gamma(x_2)/\Gamma(x_1 + x_2),\ x_1 > 0$ and $x_2 > 0$
CBETA	$\beta(z_1, z_2) = \Gamma(z_1)\Gamma(z_2)/\Gamma(z_1 + z_2),\ z_1 > 0$ and $z_2 > 0$
ALBETA	$\ln \beta(a, b),\ a > 0,\ b > 0$
CLBETA	$\ln \beta(a, b),\ \Re a > 0,\ \Re b > 0$
BETAI	$I_x(a, b) = \beta_x(a, b)/\beta(a, b),\ 0 \le x \le 1,\ a > 0,\ b > 0$

FAC/DFAC (Single/Double precision)

Evaluate the factorial of the argument.

Usage

```
FAC(N)
```

Arguments

N — Argument for which the factorial is desired. (Input)

FAC — Function value. (Output)

Comments

To evaluate the factorial for nonintegral values of the argument, the gamma function should be used. For large values of the argument, the log gamma function should be used.

Algorithm

The factorial is computed using the relation $n! = \Gamma(n + 1)$. The function $\Gamma(x)$ is defined in `GAMMA` on page 45. The argument n must be greater than or equal to zero, and it must not be so large that $n!$ overflows. Approximately, $n!$ overflows when $n^n e^{-n}$ overflows.

Example

In this example, 6! is computed and printed.

```
C                                 Declare variables
      INTEGER    N, NOUT
      REAL       FAC, VALUE
      EXTERNAL   FAC, UMACH
C                                 Compute
      N     = 6
      VALUE = FAC(N)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) N, VALUE
99999 FORMAT (' FAC(', I1, ') = ', F6.2)
      END
```

Output

```
FAC(6) = 720.00
```

BINOM/DBINOM (Single/Double precision)

Evaluate the binomial coefficient.

Usage

`BINOM(N, M)`

Arguments

N — First parameter of the binomial coefficient. (Input)
`N` must be nonnegative.

M — Second parameter of the binomial coefficient. (Input)
`M` must be nonnegative and less than or equal to `N`.

BINOM — Function value. (Output)

Comments

To evaluate binomial coefficients for nonintegral values of the arguments, the complete beta function or log beta function should be used.

Algorithm

The binomial function is defined to be

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

with $n \ge m \ge 0$. Also, n must not be so large that the function overflows.

Example

In this example, $\binom{9}{5}$ is computed and printed.

```
C                                  Declare variables
      INTEGER    M, N, NOUT
      REAL       BINOM, VALUE
      EXTERNAL   BINOM, UMACH
C                                  Compute
      N     = 9
      M     = 5
      VALUE = BINOM(N, M)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) N, M, VALUE
99999 FORMAT (' BINOM(', I1, ',', I1, ') = ', F6.2)
      END
```

Output

```
BINOM(9,5) = 126.00
```

GAMMA/DGAMMA (Single/Double precision)

Evaluate the complete gamma function.

Usage

`GAMMA(X)`

Arguments

X — Argument for which the complete gamma function is desired. (Input)

GAMMA — Function value. (Output)

Comments

Informational errors

Type	Code	
2	1	The function underflows because `X` is too small.
3	2	Result is accurate to less than one-half precision because `X` is too near a negative integer.

Algorithm

The gamma function, $\Gamma(x)$, is defined to be

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \qquad \text{for } x > 0$$

For x < 0, the above definition is extended by analytic continuation.

The gamma function is not defined for integers less than or equal to zero. Also, the argument x must be greater than $x_{\min}$ so that $\Gamma(x)$ does not underflow, and x must be less than $x_{\max}$ so that $\Gamma(x)$ does not overflow. The underflow limit occurs first for arguments that are close to large negative half integers. Even though other arguments away from these half integers may yield machine-representable values of $\Gamma(x)$, such arguments are considered illegal. Users who need such values should use the log gamma function `ALNGAM`, page 49, or `ALGAMS`, page 52. Finally, the argument should not be so close to a negative integer that the result is less accurate than half precision. The limits $x_{\min}$ and $x_{\max}$ are available by

```
CALL R9GAML (XMIN, XMAX)
CALL D9GAML (XMIN, XMAX)
```

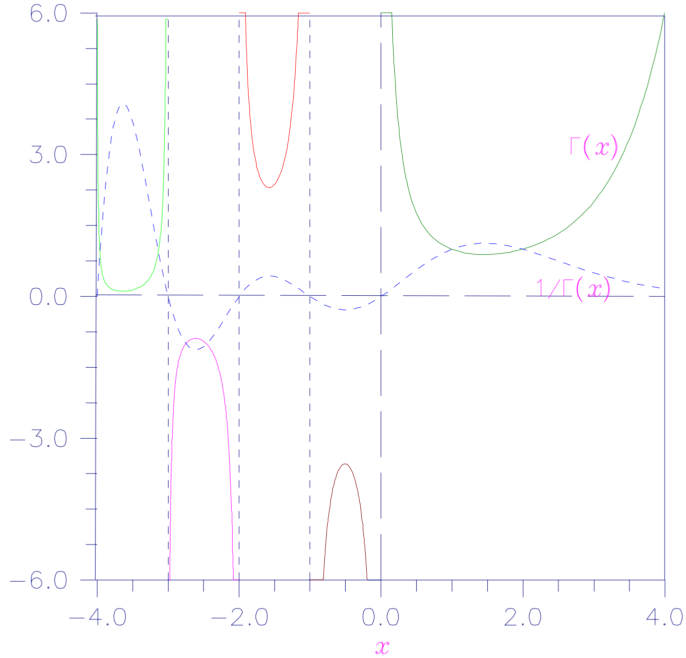

Figure 4-1 Plot of Γ(*x*) and 1/Γ(*x*)

Example

In this example, Γ(5.0) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       GAMMA, VALUE, X
      EXTERNAL   GAMMA, UMACH
C                                  Compute
      X     = 5.0
      VALUE = GAMMA(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' GAMMA(', F6.3, ') = ', F6.3)
      END
```

Output

```
GAMMA( 5.000) = 24.000
```

CGAMMA

Evaluate the complex gamma function.

Usage

`CGAMMA(Z)`

Arguments

Z — Complex argument for which the gamma function is desired. (Input)

CGAMMA — Complex function value. (Output)

Comments

This routine simply exponentiates the complex log gamma function.

Algorithm

The gamma function, $\Gamma(z)$, is defined to be

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad \text{for } \Re z > 0$$

For $\Re(z) < 0$, the above definition is extended by analytic continuation.

z must not be so close to a negative integer that the result is less accurate than half precision. If $\Re(z)$ is too small, then the result will underflow. When $\Im(z) \approx 0$, $\Re(z)$ should be greater than x_{min} so that the result does not underflow, and $\Re(z)$ should be less than x_{max} so that the result does not overflow. x_{min} and x_{max} are available by

```
CALL R9GAML (XMIN, XMAX)
CALL D9GAML (XMIN, XMAX)
```

Note that z must not be too far from the real axis because the result will underflow.

Example

In this example, $\Gamma(1.4 + 3i)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      COMPLEX    CGAMMA, VALUE, Z
      EXTERNAL   CGAMMA, UMACH
C                                  Compute
      Z     = (1.4, 3.0)
      VALUE = CGAMMA(Z)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CGAMMA(', F6.3, ',', F6.3, ') = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CGAMMA( 1.400, 3.000) = (-0.001, 0.061)
```

GAMR/DGAMR (Single/Double precision)

Evaluate the reciprocal gamma function.

Usage

`GAMR(X)`

Arguments

X — Argument for which the reciprocal gamma function is desired. (Input)

GAMR — Function value. (Output)

Algorithm

The reciprocal gamma function is defined to be $1/\Gamma(x)$. See `GAMMA` (page 45) for the definition of $\Gamma(x)$.

The gamma function is not defined for integers less than or equal to zero. Also, x must be larger than x_{min} so that $1/\Gamma(x)$ does not underflow, and x must be smaller than x_{max} so that $1/\Gamma(x)$ does not overflow. Symmetric overflow and underflow limits x_{min} and x_{max} are obtainable from

```
CALL R9GAML (XMIN, XMAX)
CALL D9GAML (XMIN, XMAX)
```

Example

In this example, $1/\Gamma(1.85)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       GAMR, VALUE, X
      EXTERNAL   GAMR, UMACH
C                                 Compute
      X     = 1.85
      VALUE = GAMR(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' GAMR(', F6.3, ') = ', F6.3)
      END
```

Output

```
GAMR( 1.850) = 1.058
```

CGAMR

Evaluate the reciprocal complex gamma function.

Usage

`CGAMR(Z)`

Arguments

Z — Complex argument for which the reciprocal gamma function is desired. (Input)

CGAMR — Complex function value. (Output)

Comments

This function is well behaved near zero and negative integers.

Algorithm

The function `CGAMR` computes $1/\Gamma(z)$. See `CGAMMA` (page 47) for the definition of $\Gamma(z)$.

For $\Im(z) \approx 0$, z must be larger than $x_{\min}$ so that $1/\Gamma(z)$ does not underflow, and x must be smaller than $x_{\max}$ so that $1/\Gamma(z)$ does not overflow. Symmetric overflow and underflow limits $x_{\min}$ and $x_{\max}$ are obtainable from

```
CALL R9GAML (XMIN, XMAX)
CALL D9GAML (XMIN, XMAX)
```

Note that z must not be too far from the real axis because the result will overflow there.

Example

In this example, $\ln \Gamma(1.4 + 3i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CGAMR, VALUE, Z
      EXTERNAL   CGAMR, UMACH
C                                 Compute
      Z     = (1.4, 3.0)
      VALUE = CGAMR(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CGAMR(', F6.3, ',', F6.3, ') = (', F7.3, ',', F7.3, ')')
      END
```

Output

```
CGAMR( 1.400, 3.000) = ( -0.303,-16.367)
```

ALNGAM/DLNGAM (Single/Double precision)

Evaluate the logarithm of the absolute value of the gamma function.

Usage

`ALNGAM(X)`

Arguments

X — Argument for which the function value is desired. (Input)

ALNGAM — Function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `ALNGAM(X)` is accurate to less than one-half precision because `X` is too near a negative integer.

Algorithm

The function `ALNGAM` computes ln $|\Gamma(x)|$. See `GAMMA` (page 45) for the definition of $\Gamma(x)$.

The gamma function is not defined for integers less than or equal to zero. Also, $|x|$ must not be so large that the result overflows. Neither should x be so close to a negative integer that the accuracy is worse than half precision.

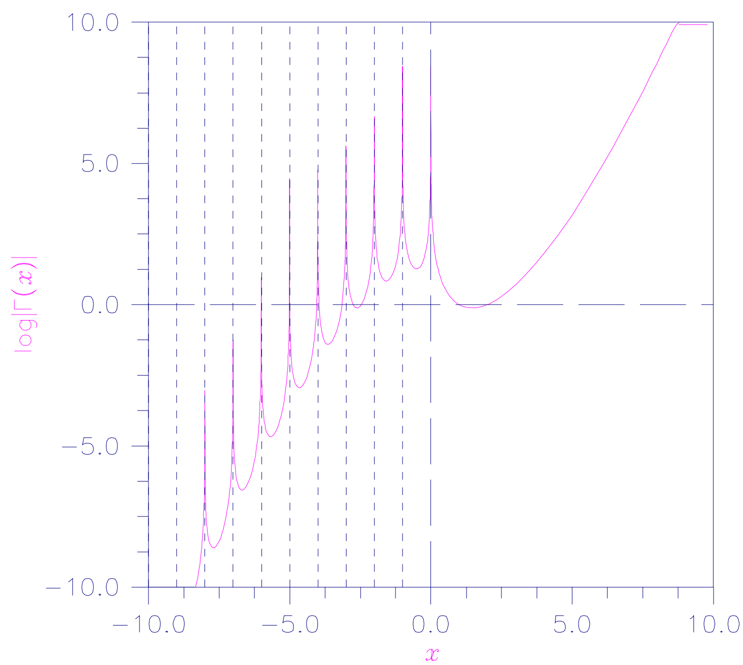

Figure 4-2 Plot of log$|\Gamma(x)|$

Example

In this example, $\ln |\Gamma(1.85)|$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ALNGAM, VALUE, X
      EXTERNAL   ALNGAM, UMACH
C                                 Compute
      X     = 1.85
      VALUE = ALNGAM(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ALNGAM(', F6.3, ') = ', F6.3)
      END
```

Output

```
ALNGAM( 1.850) = -0.056
```

CLNGAM

Evaluate the complex natural logarithm of the gamma function.

Usage

`CLNGAM(ZIN)`

Arguments

ZIN — Complex argument for which the logarithm of the gamma function is desired. (Input)

CLNGAM — Complex function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `CLNGAM(ZIN)` is accurate to less than one-half precision because `ZIN` is too near a negative integer.

Algorithm

The function `CLNGAM` computes $\ln \Gamma(z)$. See `CGAMMA` (page 47) for the definition of $\Gamma(z)$.

The argument z must not be so large that the result overflows. Neither should z be so close to a negative integer that the accuracy is worse than half precision.

Example

In this example, $\ln \Gamma(1.4 + 3i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CLNGAM, VALUE, Z
      EXTERNAL   CLNGAM, UMACH
C                                 Compute
      Z     = (1.4, 3.0)
      VALUE = CLNGAM(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CLNGAM(', F6.3, ',', F6.3, ') = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CLNGAM( 1.400, 3.000) = (-2.795, 1.589)
```

ALGAMS/DLGAMS (Single/Double precision)

Return the logarithm of the absolute value of the gamma function and the sign of gamma.

Usage

`CALL ALGAMS (X, ALGM, S)`

Arguments

X — Argument for which the logarithm of the absolute value of the gamma function is desired. (Input)

ALGM — Result of the calculation. (Output)

S — Sign of gamma(`X`). (Output)
If gamma(`X`) is greater than or equal to zero, `S` = 1.0. If gamma(`X`) is less than zero, `S` = –1.0.

Comments

Informational error

Type	Code	
3	2	Result of `ALGAMS` is accurate to less than one-half precision because `X` is too near a negative integer.

Algorithm

The function `ALGAMS` computes $\ln |\Gamma(x)|$ and the sign of $\Gamma(x)$. See `GAMMA` (page 44) for the definition of $\Gamma(x)$.

The result overflows if $|x|$ is too large. The accuracy is worse than half precision if x is too close to a negative integer.

Example

In this example, $\ln|\Gamma(1.85)|$ and the sign of $\Gamma(1.85)$ are computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, S, X
      EXTERNAL   ALGAMS, UMACH
C                                 Compute
      X = 1.85
      CALL ALGAMS(X, VALUE, S)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99998) X, VALUE
99998 FORMAT (' Log Abs(Gamma(', F6.3, ')) = ', F6.3)
      WRITE (NOUT,99999) X, S
99999 FORMAT (' Sign(Gamma(', F6.3, ')) = ', F6.2)
      END
```

Output

```
Log Abs(Gamma( 1.850)) = -0.056
Sign(Gamma( 1.850)) = 1.00
```

GAMI/DGAMI (Single/Double precision)

Evaluate the incomplete gamma function.

Usage

`GAMI(A, X)`

Arguments

A — The integrand exponent parameter. (Input)
It must be positive.

X — The upper limit of the integral definition of `GAMI`. (Input)
It must be nonnegative.

GAMI — Function value. (Output)

Algorithm

The incomplete gamma function is defined to be

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t}\, dt \qquad \text{for } a > 0 \text{ and } x \ge 0$$

The function $\gamma(a, x)$ is defined only for a greater than zero. Although $\gamma(a, x)$ is well defined for $x > -\infty$, this algorithm does not calculate $\gamma(a, x)$ for negative x. For large a and sufficiently large x, $\gamma(a, x)$ may overflow. $\gamma(a, x)$ is bounded by $\Gamma(a)$, and users may find this bound a useful guide in determining legal values of a.

Because logarithmic variables are used, a slight deterioration of two or three digits of accuracy will occur when GAMI is very large or very small.

Error! Objects cannot be created from editing field codes.

Figure 4-3 Contour Plot of $\gamma(a, x)$

Example

In this example, $\gamma(2.5, 0.9)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       A, GAMI, VALUE, X
      EXTERNAL   GAMI, UMACH
C                                 Compute
      A     = 2.5
      X     = 0.9
      VALUE = GAMI(A, X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' GAMI(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
GAMI( 2.500, 0.900) = 0.1647
```

GAMIC/DGAMIC (Single/Double precision)

Evaluate the complementary incomplete gamma function.

Usage

GAMIC(A, X)

Arguments

A — The integrand exponent parameter as per the remarks. (Input)

X — The upper limit of the integral definition of GAMIC. (Input)
If A is positive, then X must be positive. Otherwise, X must be nonnegative.

GAMIC — Function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of GAMIC(A, X) is accurate to less than one-half precision because A is too near a negative integer.

Algorithm

The incomplete gamma function is defined to be

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$$

The only general restrictions on a are that it must be positive if x is zero; otherwise, it must not be too close to a negative integer such that the accuracy of the result is less than half precision. Furthermore, $\Gamma(a, x)$ must not be so small that it underflows, or so large that it overflows. Although $\Gamma(a, x)$ is well defined for $x > -\infty$ and $a > 0$, this algorithm does not calculate $\Gamma(a, x)$ for negative x.

The function `GAMIC` is based on a code by Gautschi (1979).

Example

In this example, $\Gamma(2.5, 0.9)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       A, GAMIC, VALUE, X
      EXTERNAL   GAMIC, UMACH
C                                  Compute
      A     = 2.5
      X     = 0.9
      VALUE = GAMIC(A, X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' GAMIC(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
GAMIC( 2.500, 0.900) = 1.1646
```

GAMIT/DGAMIT (Single/Double precision)

Evaluate the Tricomi form of the incomplete gamma function.

Usage

`GAMIT(A, X)`

Arguments

A — The integrand exponent parameter as per the comments. (Input)

X — The upper limit of the integral definition of `GAMIT`. (Input)
It must be nonnegative.

GAMIT — Function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `GAMIT(A, X)` is accurate to less than one-half precision because `A` is too close to a negative integer.

Algorithm

The Tricomi's incomplete gamma function is defined to be

$$\gamma^*(a,x) = \frac{x^{-a}\gamma(a,x)}{\Gamma(a)} = \frac{x^{-a}}{\Gamma(a)}\int_x^\infty t^{a-1}e^{-t}\,dt$$

where $\gamma(a, x)$ is the incomplete gamma function. See `GAMI` (page 53) for the definition of $\gamma(a, x)$.

The only general restriction on a is that it must not be too close to a negative integer such that the accuracy of the result is less than half precision. Furthermore, $|\gamma^*(a, x)|$ must not underflow or overflow. Although $\gamma^*(a, x)$ is well defined for $x > -\infty$, this algorithm does not calculate $\gamma^*(a, x)$ for negative x.

A slight deterioration of two or three digits of accuracy will occur when `GAMIT` is very large or very small in absolute value because logarithmic variables are used. Also, if the parameter a is very close to a negative integer (but not quite a negative integer), there is a loss of accuracy which is reported if the result is less than half machine precision.

The function `GAMIT` is based on a code by Gautschi (1979).

Example

In this example, $\gamma^*(3.2, 2.1)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       A, GAMIT, VALUE, X
      EXTERNAL   GAMIT, UMACH
C                                 Compute
      A     = 3.2
      X     = 2.1
      VALUE = GAMIT(A, X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' GAMIT(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
GAMIT( 3.200, 2.100) = 0.0284
```

PSI/DPSI (Single/Double precision)

Evaluate the logarithmic derivative of the gamma function.

Usage

`PSI(X)`

Arguments

X — Argument for which the function value is desired. (Input)

PSI — Function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `PSI(X)` is accurate to less than one-half precision because `X` is too near a negative integer.

Algorithm

The psi function, also called the digamma function, is defined to be

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

See `GAMMA` (page 44) for the definition of $\Gamma(x)$.

The argument x must not be exactly zero or a negative integer, or $\psi(x)$ is undefined. Also, x must not be too close to a negative integer such that the accuracy of the result is less than half precision.

Example

In this example, $\psi(1.915)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       PSI, VALUE, X
      EXTERNAL   PSI, UMACH
C                                 Compute
      X     = 1.915
      VALUE = PSI(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' PSI(', F6.3, ') = ', F6.3)
      END
```

Output

```
PSI( 1.915) = 0.366
```

CPSI

Evaluate the logarithmic derivative of the gamma function for a complex argument.

Usage

`CPSI(ZIN)`

Arguments

ZIN — Complex argument for which the logarithmic derivative of the gamma function is desired. (Input)

CPSI — Complex function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `CPSI(ZIN)` is accurate to less than one-half precision because the argument is too near a negative integer.

Algorithm

The psi function, also called the digamma function, is defined to be

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

See `CGAMMA` (page 46) for the definition of $\Gamma(z)$.

The argument $|z|$ must not be so small that $1/z$ and therefore $\psi(z)$ overflows. If z is close to a negative integer, the result is less accurate than half precision. If z is exactly a negative integer, the result is undefined.

Example

In this example, $\psi(1.9 + 4.3i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CPSI, VALUE, Z
      EXTERNAL   CPSI, UMACH
C                                 Compute
      Z     = (1.9, 4.3)
      VALUE = CPSI(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CPSI(', F6.3, ',', F6.3, ') = (', F6.3, ',', F6.3, ')')
      END
```

Output

```
CPSI( 1.900, 4.300) = ( 1.507, 1.255)
```

POCH/DPOCH (Single/Double precision)

Evaluate a generalization of Pochhammer's symbol.

Usage

`POCH(A, X)`

Arguments

A — The first argument. (Input)

X — The second, differential argument. (Input)

POCH — Function value. (Output)
The generalized Pochhammer symbol is $\Gamma(a + x)/\Gamma(a)$.

Comments

1. Informational errors

Type	Code	
3	2	Result of `POCH(A, X)` is accurate to less than one-half precision because the absolute value of the `X` is too large. Therefore, `A + X` cannot be evaluated accurately.
3	2	Result of `POCH(A, X)` is accurate to less than one-half precision because either `A` or `A + X` is too close to a negative integer.

2. For `X` a nonnegative integer, `POCH(A, X)` is just Pochhammer's symbol.

Algorithm

Pochhammer's symbol is $(a)_n = (a)(a - 1)\ldots(a - n + 1)$ for n a nonnegative integer. Pochhammer's generalized symbol is defined to be

$$(a)_x = \frac{\Gamma(a+x)}{\Gamma(a)}$$

See `GAMMA` (page 44) for the definition of $\Gamma(x)$.

Note that a straightforward evaluation of Pochhammer's generalized symbol with either gamma or log gamma functions can be especially unreliable when a is large or x is small.

Substantial loss can occur if $a + x$ or a are close to a negative integer unless $|x|$ is sufficiently small. To insure that the result does not overflow or underflow, one can keep the arguments a and $a + x$ well within the range dictated by the gamma function routine `GAMMA` or one can keep $|x|$ small whenever a is large. `POCH` also works for a variety of arguments outside these rough limits, but any more general limits that are also useful are difficult to specify.

Example

In this example, $(1.6)_{0.8}$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       A, POCH, VALUE, X
      EXTERNAL   POCH, UMACH
C                                 Compute
      A     = 1.6
      X     = 0.8
      VALUE = POCH(A, X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' POCH(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
POCH( 1.600, 0.800) = 1.3902
```

POCH1/DPOCH1 (Single/Double precision)

Evaluate a generalization of Pochhammer's symbol starting from the first order.

Usage

`POCH1(A, X)`

Arguments

A — The first argument. (Input)

X — The second, differential argument. (Input)

POCH1 — Function value. (Output)
`POCH1(A, X)` = (`POCH(A, X)` – 1)/`X`.

Algorithm

Pochhammer's symbol from the first order is defined to be

$$\mathrm{POCH1}(a, x) = \frac{(a)_x - 1}{x} = \frac{\Gamma(a+x)}{\Gamma(a) - 1} / x$$

where $(a)_x$ is Pochhammer's generalized symbol. See `POCH` (page 59) for the definition of $(a)_x$. It is useful in special situations that require especially accurate values when x is small. This specification is particularly suited for stability when computing expressions such as

$$\left[\frac{\Gamma(a+x)}{\Gamma(a)}-\frac{\Gamma(b+x)}{\Gamma(b)}\right]/\,x=\mathrm{POCH1}(a,x)-\mathrm{POCH1}(b,x)$$

Note that `POCH1`(a, 0) = $\psi(a)$. See `PSI` (page 57) for the definition of $\psi(a)$.

When $|x|$ is so small that substantial cancellation will occur if the straightforward formula is used, we use an expansion due to fields and discussed by Luke (1969).

The ratio $(a)_x = \Gamma(a + x)/\Gamma(a)$ is written by Luke as $(a + (x - 1)/2)^x$ times a polynomial in $(a + (x - 1)/2)^{-2}$. To maintain significance in `POCH1`, we write for positive a.

$$(a + (x - 1)/2)^x = \exp(x \ln(a + (x - 1)/2)) = e^q = 1 + q\texttt{EXPRL}(q)$$

where `EXPRL` = $(e^x - 1)/x$. Likewise, the polynomial is written $P = 1 + xP_1(a, x)$. Thus,

$$\texttt{POCH1}\,(a, x) = ((a)_x - 1)/x = \texttt{EXPRL}(q)(q/x + qP_1(a, x)) + P_1(a, x)$$

Substantial significance loss can occur if $a + x$ or a are close to a negative integer even when $|x|$ is very small. To insure that the result does not overflow or underflow, one can keep the arguments a and $a + x$ well within the range dictated by the gamma function routine `GAMMA` (page 44) or one can keep $|x|$ small whenever a is large. `POCH` also works for a variety of arguments outside these rough limits, but any more general limits that are also useful are difficult to specify.

Example

In this example, `POCH1`(1.6, 0.8) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       A, POCH1, VALUE, X
      EXTERNAL   POCH1, UMACH
C                                  Compute
      A     = 1.6
      X     = 0.8
      VALUE = POCH1(A, X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' POCH1(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
POCH1( 1.600, 0.800) = 0.4878
```

BETA/DBETA (Single/Double precision)

Evaluate the complete beta function.

Usage

`BETA(A, B)`

Arguments

A — First beta parameter. (Input)
It must be positive.

B — Second beta parameter. (Input)
It must be positive.

BETA — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because A and/or B is too large.

Algorithm

The beta function is defined to be

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$$

See GAMMA (page 44) for the definition of $\Gamma(x)$.

The function BETA requires that both arguments be positive. In addition, the arguments must not be so large that the result underflows.

Example

In this example, $\beta(2.2, 3.7)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       A, BETA, VALUE, X
      EXTERNAL   BETA, UMACH
C                                  Compute
      A     = 2.2
      X     = 3.7
      VALUE = BETA(A, X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' BETA(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
BETA( 2.200, 3.700) = 0.0454
```

CBETA

Evaluate the complex complete beta function.

Usage

CBETA(A, B)

Arguments

A — Complex first beta distribution parameter. (Input)
It must be positive.

B — Complex second beta distribution parameter. (Input)
It must be positive.

CBETA — Complex function value. (Output)

Algorithm

The beta function is defined to be

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$$

See `CGAMMA` (page 46) for the definition of $\Gamma(z)$.

The arguments a and $a + b$ must not be close to negative integers. The arguments should not be so large (near the real axis) that the result underflows. Also, $a + b$ should not be so far from the real axis that the result overflows.

Example

In this example, $\beta(1.7 + 2.2i, 3.7 + 0.4i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    A, B, CBETA, VALUE
      EXTERNAL   CBETA, UMACH
C                                 Compute
      A     = (1.7, 2.2)
      B     = (3.7, 0.4)
      VALUE = CBETA(A, B)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, B, VALUE
99999 FORMAT (' CBETA((', F6.3, ',', F6.3, '), (', F6.3, ',', F6.3,
     &     ')) = (', F6.3, ',', F6.3, ')')
      END
```

Output

```
CBETA(( 1.700, 2.200), ( 3.700, 0.400)) = (-0.033,-0.017)
```

ALBETA/DLBETA (Single/Double precision)

Evaluate the natural logarithm of the complete beta function for positive arguments.

Usage

`ALBETA(A, B)`

Arguments

A — The first argument of the `BETA` function. (Input)
It must be greater than zero.

B — The second argument of the BETA function. (Input)
It must be greater than zero.

ALBETA — Function value. (Output)
ALBETA returns $\ln \beta(A, B) = \ln(\Gamma(A)\Gamma(B))/\Gamma(A + B)$.

Comments

Note that $\ln \beta(A, B) = \ln \beta(B, A)$.

Algorithm

ALBETA computes $\ln \beta(a, b) = \ln \beta(b, a)$. See BETA (page 62) for the definition of $\beta(a, b)$.

The function ALBETA is defined for $a > 0$ and $b > 0$. It returns accurate results even when a or b is very small. It can overflow for very large arguments; this error condition is not detected except by the computer hardware.

Example

In this example, $\ln \beta(2.2, 3.7)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       A, ALBETA, VALUE, X
      EXTERNAL   ALBETA, UMACH
C                                  Compute
      A     = 2.2
      X     = 3.7
      VALUE = ALBETA(A, X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' ALBETA(', F6.3, ',', F6.3, ') = ', F8.4)
      END
```

Output

```
ALBETA( 2.200, 3.700) = -3.0928
```

CLBETA

Evaluate the complex logarithm of the complete beta function.

Usage

CLBETA(A, B)

Arguments

A — Complex first beta distribution parameter. (Input)

B — Complex second beta distribution parameter. (Input)

CLBETA — Complex function value. (Output)

Algorithm

The function `CLBETA` computes ln β(*a*, *b*). See `CBETA` (page 63) for the definition of β(*a*, *b*).

The arguments *a*, *b* and *a* + *b* must not be close to negative integers (even though some combinations ought to be allowed). The arguments should not be so large that the logarithm of the gamma function overflows (presumably an improbable condition).

Example

In this example, ln β(1.7 + 2.2*i*, 3.7 + 0.4*i*) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    A, B, CLBETA, VALUE
      EXTERNAL   CLBETA, UMACH
C                                 Compute
      A     = (1.7, 2.2)
      B     = (3.7, 0.4)
      VALUE = CLBETA(A, B)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, B, VALUE
99999 FORMAT (' CLBETA((', F6.3, ',', F6.3, '), (', F6.3, ',', F6.3,
     &     ')) = (', F6.3, ',', F6.3, ')')
      END
```

Output

```
CLBETA(( 1.700, 2.200), ( 3.700, 0.400)) = (-3.280,-2.659)
```

BETAI/DBETAI (Single/Double precision)

Evaluate the incomplete beta function ratio.

Usage

`BETAI(X, PIN, QIN)`

Arguments

X — Upper limit of integration. (Input)
`X` must be in the interval (0.0, 1.0) inclusive.

PIN — First beta distribution parameter. (Input)
`PIN` must be positive.

QIN — Second beta distribution parameter. (Input)
`QIN` must be positive.

BETAI — Probability that a random variable from a beta distribution having parameters `PIN` and `QIN` will be less than or equal to `X`. (Output)

Algorithm

The incomplete beta function is defined to be

$$I_x(p, q) = \frac{\beta_x(p, q)}{\beta(p, q)} = \frac{1}{\beta(p, q)} \int_0^x t^{p-1}(1-t)^{q-1}\,dt$$
$$\text{for } 0 \le x \le 1, p > 0, q > 0$$

See `BETA` (page 62) for the definition of $\beta(p, q)$.

The parameters p and q must both be greater than zero. The argument x must lie in the range 0 to 1. The incomplete beta function can underflow for sufficiently small x and large p; however, this underflow is not reported as an error. Instead, the value zero is returned as the function value.

The function `BETAI` is based on the work of Bosten and Battiste (1974).

Example

In this example, $I_{0.61}(2.2, 3.7)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BETAI, PIN, QIN, VALUE, X
      EXTERNAL   BETAI, UMACH
C                                 Compute
      X     = 0.61
      PIN   = 2.2
      QIN   = 3.7
      VALUE = BETAI(X, PIN, QIN)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, PIN, QIN, VALUE
99999 FORMAT (' BETAI(', F6.3, ',', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
BETAI( 0.610, 2.200, 3.700) = 0.8822
```

Chapter 5: Error Function and Related Functions

Routines

5.1. Error Functions

Evaluate the error function, erf *x* .... ERF 70

Evaluate the complementary error function, erfc *x* .... ERFC 71

Evaluate the scaled complementary error function, e^{x^2} erfc x .... ERFCE 73

Evaluate a scaled function related to erfc, e^{-z^2} erfc $(-iz)$ .... CERFE 75

Evaluate the inverse error function, $\mathrm{erf}^{-1}\ x$ .... ERFI 76

Evaluate the inverse complementary error function, $\mathrm{erfc}^{-1}\ x$ .... ERFCI 77

Evaluate Dawson's function .... DAWS 79

5.2. Fresnel Integrals

Evaluate the cosine Fresnel integral, $C(x)$ .... FRESC 81

Evaluate the sine Fresnel integral, $S(x)$ .... FRESS 81

Usage Notes

The error function is

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

The complementary error function is $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$. Dawson's function is

$$e^{-x^2} \int_0^x e^{t^2}\, dt$$

The Fresnel integrals are

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt$$

and

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt$$

They are related to the error function by

$$C(z) + iS(z) = \frac{1+i}{2}\operatorname{erf}\left(\frac{\sqrt{\pi}}{2}(1-i)z\right)$$

ERF/DERF (Single/Double precision)

Evaluate the error function.

Usage

`ERF(X)`

Arguments

X — Argument for which the function value is desired. (Input)

ERF — Function value. (Output)

Algorithm

The error function, erf(x), is defined to be

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$$

All values of x are legal.

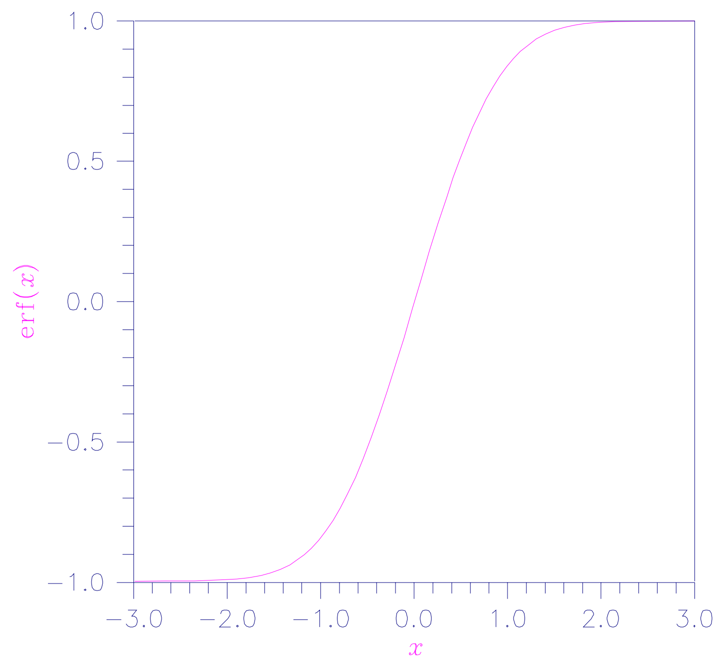

Figure 5-1   Plot of erf *x*

Example

In this example, erf(1.0) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ERF, VALUE, X
      EXTERNAL   ERF, UMACH
C                                 Compute
      X     = 1.0
      VALUE = ERF(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ERF(', F6.3, ') = ', F6.3)
      END
```

Output

```
ERF( 1.000) = 0.843
```

ERFC/DERFC (Single/Double precision)

Evaluate the complementary error function.

Usage

`ERFC(X)`

Arguments

X — Argument for which the function value is desired. (Input)

ERFC — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too large.

Algorithm

The complementary error function, erfc(x), is defined to be

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$

The argument x must not be so large that the result underflows. Approximately, x should be less than

$$\left[-\ln\left(\sqrt{\pi}s\right)\right]^{1/2}$$

where s = `AMACH`(1) (page 240) is the smallest representable positive floating-point number.

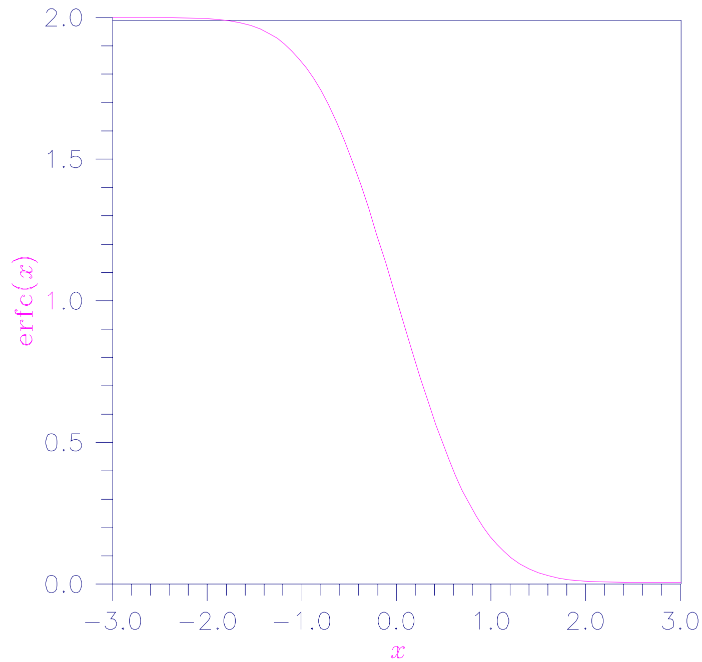

Figure 5-2 Plot of erfc *x*

Example

In this example, erfc(1.0) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ERFC, VALUE, X
      EXTERNAL   ERFC, UMACH
C                                 Compute
      X     = 1.0
      VALUE = ERFC(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ERFC(', F6.3, ') = ', F6.3)
      END
```

Output

```
ERFC( 1.000) = 0.157
```

ERFCE/DERFCE (Single/Double precision)

Evaluate the exponentially scaled complementary error function.

Usage

`ERFCE(X)`

Arguments

X — Argument for which the function value is desired. (Input)

ERFCE — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too large.

Algorithm

The function `ERFCE(X)` computes

$$e^{x^2} \operatorname{erfc}(x)$$

where erfc(x) is the complementary error function. See `ERFC` (page 71) for its definition.

To prevent the answer from underflowing, x must be greater than

$$x_{\min} \simeq -\sqrt{\ln(b/2)}$$

where b = `AMACH`(2) is the largest representable floating-point number.

Example

In this example, `ERFCE`(1.0) = $e^{1.0}$ erfc(1.0) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ERFCE, VALUE, X
      EXTERNAL   ERFCE, UMACH
C                                 Compute
      X     = 1.0
      VALUE = ERFCE(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ERFCE(', F6.3, ') = ', F6.3)
      END
```

Output

```
ERFCE( 1.000) = 0.428
```

CERFE/ZERFE (Single/Double precision)

Evaluate the complex scaled complemented error function.

Usage

`CERFE(Z)`

Arguments

Z — Complex argument for which the function value is desired. (Input)

CERFE — Complex function value. (Output)

Algorithm

Function `CERFCE` is defined to be

$$e^{-z^2}\operatorname{erfc}(-iz) = -ie^{-z^2}\frac{2}{\sqrt{\pi}}\int_z^\infty e^{t^2}\,dt$$

Let b = `AMACH`(2) be the largest floating-point number. The argument z must satisfy

$$|z| \le \sqrt{b}$$

or else the value returned is zero. If the argument z does not satisfy $(\Im z)^2 - (\Re z)^2 \le \log b$, then b is returned. All other arguments are legal (Gautschi 1969, 1970).

Example

In this example, `CERFE`$(2.5 + 2.5i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CERFE, VALUE, Z
      EXTERNAL   CERFE, UMACH
C                                 Compute
      Z     = (2.5, 2.5)
      VALUE = CERFE(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CERFE(', F6.3, ',', F6.3, ') = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CERFE( 2.500, 2.500) = ( 0.117, 0.108)
```

ERFI/DERFI (Single/Double precision)

Evaluate the inverse error function.

Usage

`ERFI(X)`

Arguments

X — Argument for which the function value is desired. (Input)

ERFI — Function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `ERFI(X)` is accurate to less than one-half precision because the absolute value of the argument is too large.

Algorithm

Function `ERFI(X)` computes the inverse of the error function erf x, defined in `ERF` (page 70).

The function `ERFI(X)` is defined for $|x| < 1$. If $x_{max} < |x| < 1$, then the answer will be less accurate than half precision. Very approximately,

$$x_{\max} \approx 1 - \sqrt{\varepsilon / (4\pi)}$$

where ε = `AMACH(4)` is the machine precision.

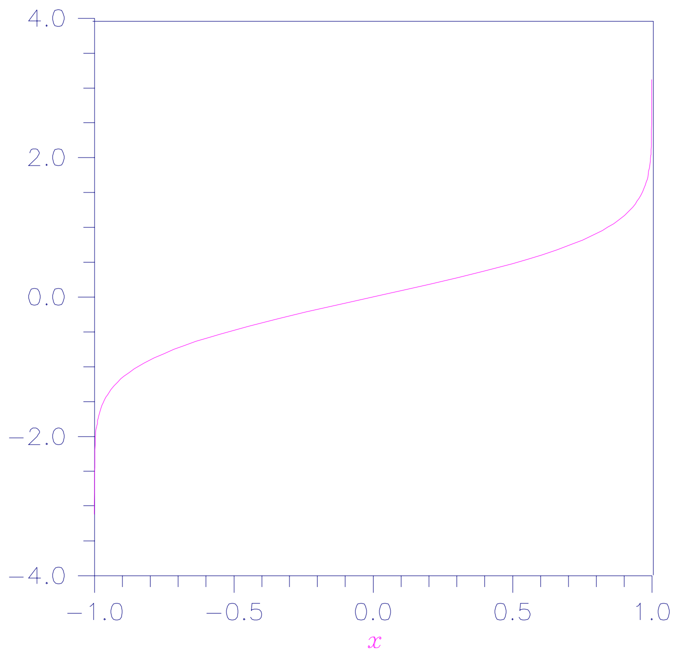

Figure 5-3 Plot of $\mathrm{erf}^{-1} x$

Example

In this example, $\mathrm{erf}^{-1}(\mathrm{erf}(1.0))$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       ERF, ERFI, VALUE, X
      EXTERNAL   ERF, ERFI, UMACH
C                                  Compute
      X     = ERF(1.0)
      VALUE = ERFI(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ERFI(', F6.3, ') = ', F6.3)
      END
```

Output

```
ERFI( 0.843) = 1.000
```

ERFCI/DERFCI (Single/Double precision)

Evaluate the inverse complementary error function.

Usage

`ERFCI(X)`

Arguments

X — Argument for which the function value is desired. (Input)

ERFCI — Function value. (Output)

Comments

Informational error

Type	Code	
3	2	Result of `ERFCI(X)` is accurate to less than one-half precision because the argument is too close to 2.0.

Algorithm

The function `ERFCI(X)` computes the inverse of the complementary error function erfc x, defined in `ERFC` (page 71).

The function `ERFCI(X)` is defined for $0 < x < 2$. If $x_{max} < x < 2$, then the answer will be less accurate than half precision. Very approximately,

$$x_{\max} \approx 2 - \sqrt{\varepsilon / (4\pi)}$$

where ε = `AMACH(4)` is the machine precision.

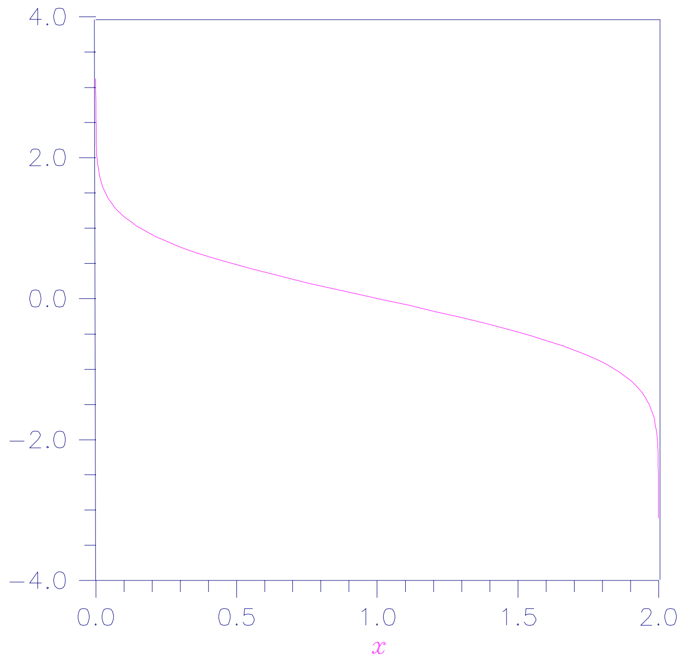

Figure 5-4 Plot of $\mathrm{erf}^{-1} x$

Example

In this example, $\mathrm{erfc}^{-1}(\mathrm{erfc}(1.0))$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ERFC, ERFCI, VALUE, X
      EXTERNAL   ERFC, ERFCI, UMACH
C                                 Compute
      X     = ERFC(1.0)
      VALUE = ERFCI(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ERFCI(', F6.3, ') = ', F6.3)
      END
```

Output

```
ERFCI( 0.157) = 1.000
```

DAWS/DDAWS (Single/Double precision)

Evaluate Dawson's function.

Usage

`DAWS(X)`

Arguments

X — Argument for which the function value is desired. (Input)

DAWS — Function value. (Output)

Comments

1. Informational error

Type	Code	
2	1	The function underflows because the absolute value of `X` is too large.

2. The Dawson function is closely related to the error function for imaginary arguments.

Algorithm

Dawson's function is defined to be

$$e^{-x^2} \int_0^x e^{t^2} dt$$

It is closely related to the error function for imaginary arguments.

So that Dawson's function does not underflow, $|x|$ must be less than $1/(2s)$. Here, s = `AMACH`(1) is the smallest representable positive floating-point number.

Example

In this example, `DAWS`(1.0) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       DAWS, VALUE, X
      EXTERNAL   DAWS, UMACH
C                                 Compute
      X     = 1.0
      VALUE = DAWS(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' DAWS(', F6.3, ') = ', F6.3)
      END
```

Output

```
DAWS( 1.000) = 0.538
```

FRESC/DFRESC (Single/Double precision)

Evaluate the cosine Fresnel integral.

Usage

`FRESC(X)`

Arguments

X — Argument for which the function value is desired. (Input)

FRESC — Function value. (Output)

Algorithm

The cosine Fresnel integral is defined to be

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt$$

All values of x are legal.

Example

In this example, $C(1.75)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       FRESC, VALUE, X
      EXTERNAL   FRESC, UMACH
C                                 Compute
      X     = 1.75
      VALUE = FRESC(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' FRESC(', F6.3, ') = ', F6.3)
      END
```

Output

```
FRESC( 1.750) = 0.322
```

FRESS/DFRESS (Single/Double precision)

Evaluate the sine Fresnel integral.

Usage

`FRESS(X)`

Arguments

X — Argument for which the function value is desired. (Input)

FRESS — Function value. (Output)

Algorithm

The sine Fresnel integral is defined to be

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$$

All values of x are legal.

Example

In this example, $S(1.75)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       FRESS, VALUE, X
      EXTERNAL   FRESS, UMACH
C                                 Compute
      X     = 1.75
      VALUE = FRESS(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' FRESS(', F6.3, ') = ', F6.3)
      END
```

Output

```
FRESS( 1.750) = 0.499
```

Chapter 6: Bessel Functions

Routines

6.1. Bessel Functions of Order 0 and 1

Evaluate $J_0(x)$ BSJ0 84
Evaluate $J_1(x)$ BSJ1 86
Evaluate $Y_0(x)$ BSY0 87
Evaluate $Y_1(x)$ BSY1 88
Evaluate $I_0(x)$ BSI0 89
Evaluate $I_1(x)$ BSI1 91
Evaluate $K_0(x)$ BSK0 92
Evaluate $K_1(x)$ BSK1 93
Evaluate $e^{-|x|}I_0(x)$ BSI0E 95
Evaluate $e^{-|x|}I_1(x)$ BSI1E 95
Evaluate $e^{x}K_0(x)$ BSK0E 96
Evaluate $e^{x}K_1(x)$ BSK1E 97

6.2. Series of Bessel Functions, Integer Order

Evaluate $J_k(x)$, $k = 0, \ldots, n - 1$ .... BSJNS 98
Evaluate $J_k(z)$, $k = 0, \ldots, n - 1$, z complex .... CBJNS 99
Evaluate $I_k(x)$, $k = 0, \ldots, n - 1$ .... BSINS 100
Evaluate $I_k(z)$, $k = 0, \ldots, n - 1$, z complex .... CBINS 102

6.3. Series of Bessel Functions, Real Order and Argument

Evaluate $J_{\nu+k}(x)$, $k = 0, \ldots, n - 1$ .... BSJS 103
Evaluate $Y_{\nu+k}(x)$, $k = 0, \ldots, n - 1$ .... BSYS 105
Evaluate $I_{\nu+k}(x)$, $k = 0, \ldots, n - 1$ .... BSIS 106
Evaluate $e^{-x}I_{\nu+k}(x)$, $k = 0, \ldots, n - 1$ .... BSIES 107
Evaluate $K_{\nu+k}(x)$, $k = 0, \ldots, n - 1$ .... BSKS 109
Evaluate $e^{x}K_{\nu+k}(x)$, $k = 0, \ldots, n - 1$ .... BSKES 110

6.4. **Series of Bessel Functions, Real Order and Complex Argument**
Evaluate $J_{\nu+k}(z)$, $k = 0, \ldots, n-1$ .... CBJS 112
Evaluate $Y_{\nu+k}(z)$, $k = 0, \ldots, n-1$ .... CBYS 113
Evaluate $I_{\nu+k}(z)$, $k = 0, \ldots, n-1$ .... CBIS 115
Evaluate $K_{\nu+k}(z)$, $k = 0, \ldots, n-1$ .... CBKS 117

Usage Notes

The following table lists the Bessel function routines by argument and order type:

	Real Argument				Complex Argument	
	Order				Order	
Function	**0**	**1**	**integer**	**real**	**integer**	**real**
$J_\nu(x)$	BSJ0 p. 84	BSJ1 p. 86	BSJNS p. 98	BSJS p. 103	CBJNS p. 99	CBJS p. 112
$Y_\nu(x)$	BSY0 p. 87	BSY1 p. 88		BSYS p. 105		CBYS p. 113
$I_\nu(x)$	BSI0 p. 89	BSI1 p. 91	BSINS p. 100	BSIS p. 106	CBINS p. 102	CBIS p. 115
$e^{-\|x\|}I_\nu(x)$	BSI0E p. 95	BSI1E p. 95		BSIES p. 107		
$K_\nu(x)$	BSK0 p. 92	BSK1 p. 93		BSKS p. 109		CBKS p. 117
$e^{-\|x\|}K_\nu(x)$	BSK0E p. 96	BSK1E p. 97		BSKES p. 110		

BSJ0/DBSJ0 (Single/Double precision)

Evaluate the Bessel function of the first kind of order zero.

Usage

```
BSJ0(X)
```

Arguments

X — Argument for which the function value is desired. (Input)

BSJ0 — Function value. (Output)

Algorithm

The Bessel function $J_0(x)$ is defined to be

$$J_0(x) = \frac{1}{\pi}\int_0^n \cos(x \sin\theta)\, d\theta$$

To prevent the answer from being less accurate than half precision, $|x|$ should be smaller than

$$1/\sqrt{\varepsilon}$$

For the result to have any precision at all, $|x|$ must be less than $1/\varepsilon$. Here, ε is the machine precision, ε = AMACH(4).

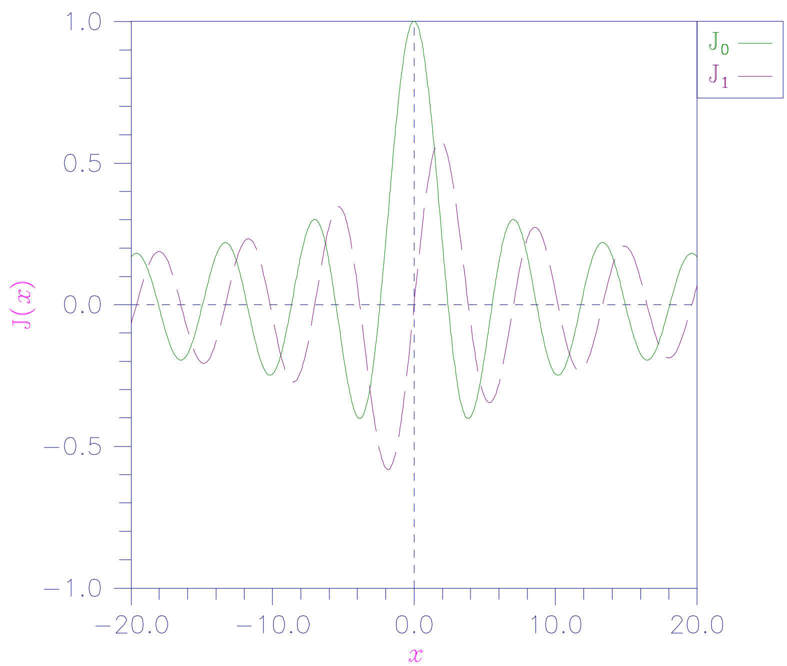

Figure 6-1 Plot of $J_0(x)$ and $J_1(x)$

Example

In this example, $J_0(3.0)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BSJ0, VALUE, X
      EXTERNAL   BSJ0, UMACH
C                                 Compute
      X     = 3.0
      VALUE = BSJ0(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSJ0(', F6.3, ') = ', F6.3)
```

```
      END
```

Output

```
BSJ0( 3.000) = -0.260
```

BSJ1/DBSJ1 (Single/Double precision)

Evaluate the Bessel function of the first kind of order one.

Usage

`BSJ1(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BSJ1 — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because the absolute value of `X` is too small.

Algorithm

The Bessel function $J_1(x)$ is defined to be

$$J_1(x) = \frac{1}{\pi}\int_0^n \cos(\theta - x\sin\theta)\,d\,\theta$$

The argument x must be zero or larger in absolute value than $2s$ to prevent $J_1(x)$ from underflowing. Also, $|x|$ should be smaller than

$$1 / \sqrt{\varepsilon}$$

to prevent the answer from being less accurate than half precision. $|x|$ must be less than $1/\varepsilon$ for the result to have any precision at all. Here, ε is the machine precision, ε = `AMACH`(4), and s = `AMACH`(1) is the smallest representable positive floating-point number.

Example

In this example, $J_1(2.5)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BSJ1, VALUE, X
      EXTERNAL   BSJ1, UMACH
C                                 Compute
```

```
      X     = 2.5
      VALUE = BSJ1(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSJ1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSJ1( 2.500) = 0.497
```

BSY0/DBSY0 (Single/Double precision)

Evaluate the Bessel function of the second kind of order zero.

Usage

BSY0(X)

Arguments

X — Argument for which the function value is desired. (Input)

BSY0 — Function value. (Output)

Algorithm

The Bessel function $Y_0(x)$ is defined to be

$$Y_0(x) = \frac{1}{\pi}\int_0^n \sin(x \sin\theta)\, d\theta$$

To prevent the answer from being less accurate than half precision, x should be smaller than

$$1/\sqrt{\varepsilon}$$

For the result to have any precision at all, $|x|$ must be less than $1/\varepsilon$. Here, ε is the machine precision, ε = AMACH(4).

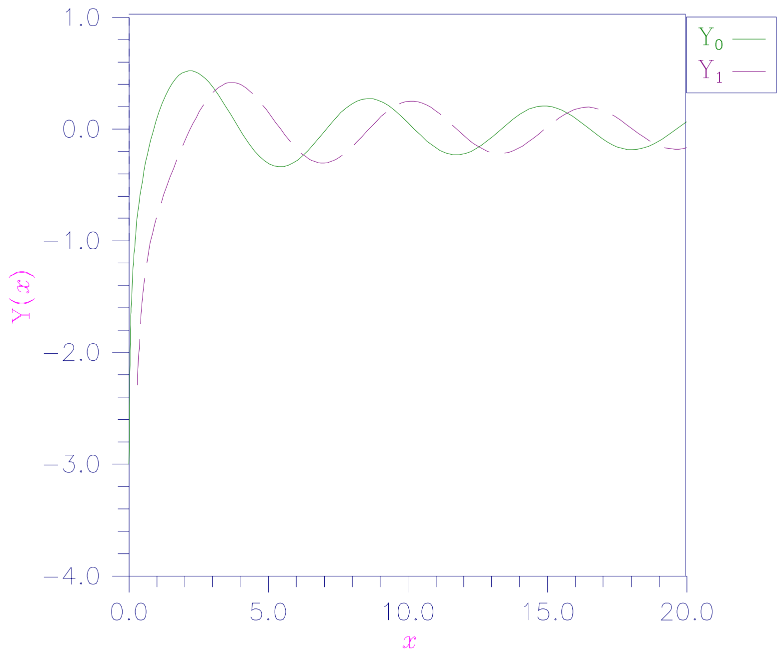

Figure 6-2   Plot of $Y_0(x)$ and $Y_1(x)$

Example

In this example, $Y_0(3.0)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BSY0, VALUE, X
      EXTERNAL   BSY0, UMACH
C                                 Compute
      X     = 3.0
      VALUE = BSY0(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSY0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSY0( 3.000) = 0.377
```

BSY1/DBSY1 (Single/Double precision)

Evaluate the Bessel function of the second kind of order one.

Usage

`BSY1(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BSY1 — Function value. (Output)

Algorithm

The Bessel function $Y_1(x)$ is defined to be

$$Y_1(x) = \frac{1}{\pi}\int_0^\pi \sin(\theta - x\sin\theta)\,d\theta$$

$Y_1(x)$ is defined for $x > 0$. To prevent the answer from being less accurate than half precision, x should be smaller than

$$1/\sqrt{\varepsilon}$$

For the result to have any precision at all, $|x|$ must be less than $1/\varepsilon$. Here, ε is the machine precision, ε = `AMACH`(4).

Example

In this example, $Y_1(3.0)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       BSY1, VALUE, X
      EXTERNAL   BSY1, UMACH
C                                  Compute
      X     = 3.0
      VALUE = BSY1(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSY1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSY1( 3.000) = 0.325
```

BSI0/DBSI0 (Single/Double precision)

Evaluate the modified Bessel function of the first kind of order zero.

Usage

`BSI0(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BSI0 — Function value. (Output)

Algorithm

The Bessel function $I_0(x)$ is defined to be

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)\,d\theta$$

The absolute value of the argument x must not be so large that $e^{|x|}$ overflows.

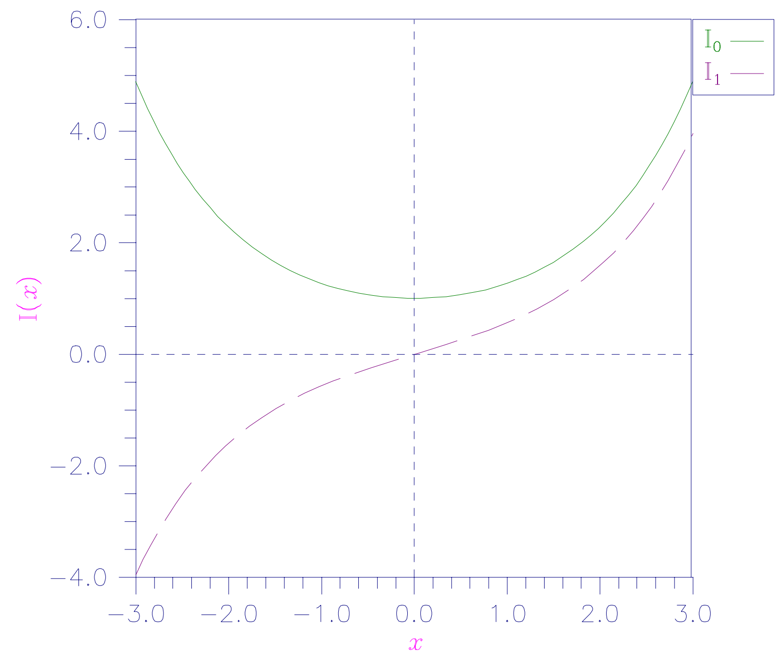

Figure 6-3 Plot of $I_0(x)$ and $I_1(x)$

Example

In this example, $I_0(4.5)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BSI0, VALUE, X
      EXTERNAL   BSI0, UMACH
C                                 Compute
      X     = 4.5
      VALUE = BSI0(X)
C                                 Print the results
```

```
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSI0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSI0( 4.500) = 17.481
```

BSI1/DBSI1 (Single/Double precision)

Evaluate the modified Bessel function of the first kind of order one.

Usage

`BSI1(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BSI1 — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because the absolute value of `X` is too small.

Algorithm

The Bessel function $I_1(x)$ is defined to be

$$I_1(x) = \frac{1}{\pi}\int_0^{\pi} \exp(x\cos\theta)\cos\theta\, d\theta$$

The argument should not be so close to zero that $I_1(x) \approx x/2$ underflows, nor so large in absolute value that $e^{|x|}$ and, therefore, $I_1(x)$ overflows.

Example

In this example, $I_1(4.5)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BSI1, VALUE, X
      EXTERNAL   BSI1, UMACH
C                                 Compute
      X     = 4.5
      VALUE = BSI1(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
```

```
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSI1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSI1( 4.500) = 15.389
```

BSK0/DBSK0 (Single/Double precision)

Evaluate the modified Bessel function of the third kind of order zero.

Usage

`BSK0(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BSK0 — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too large.

Algorithm

The Bessel function $K_0(x)$ is defined to be

$$K_0(x) = \int_0^{\infty} \cos(x \sin t)\, dt$$

The argument must be larger than zero, but not so large that the result, approximately equal to

$$\sqrt{\pi / (2x)}\, e^{-x}$$

underflows.

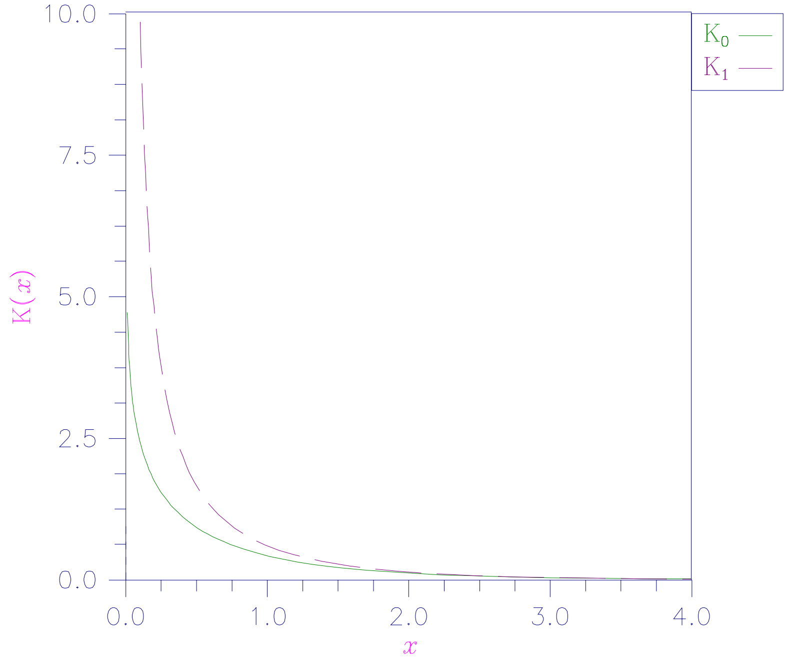

Figure 6-4 Plot of $K_0(x)$ and $K_1(x)$

Example

In this example, $K_0(0.5)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BSK0, VALUE, X
      EXTERNAL   BSK0, UMACH
C                                 Compute
      X     = 0.5
      VALUE = BSK0(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSK0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSK0( 0.500) = 0.924
```

BSK1/DBSK1 (Single/Double precision)

Evaluate the modified Bessel function of the third kind of order one.

Usage

`BSK1(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BSK1 — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too large.

Algorithm

The Bessel function $K_1(x)$ is defined to be

$$K_1(x) = \int_0^\infty \sin(x \sin t) \sin t \ dt$$

The argument x must be large enough ($> \max(1/b, s)$) that $K_1(x)$ does not overflow, and x must be small enough that the approximate answer,

$$\sqrt{\pi / (2x)}\, e^{-x}$$

does not underflow. Here, s is the smallest representable positive floating-point number, s = `AMACH`(1) , and b = `AMACH`(2) is the largest representable floating-point number.

Example

In this example, $K_1(0.5)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BSK1, VALUE, X
      EXTERNAL   BSK1, UMACH
C                                 Compute
      X     = 0.5
      VALUE = BSK1(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSK1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSK1( 0.500) = 1.656
```

BSI0E/DBSI0E (Single/Double precision)

Evaluate the exponentially scaled modified Bessel function of the first kind of order zero.

Usage

`BSI0E(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BSI0E — Function value. (Output)

Algorithm

Function `BSI0E` computes $e^{-|x|} I_0(x)$. For the definition of the Bessel function $I_0(x)$, see `BSI0` (page 89).

Example

In this example, `BSI0E`(4.5) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BSI0E, VALUE, X
      EXTERNAL   BSI0E, UMACH
C                                 Compute
      X     = 4.5
      VALUE = BSI0E(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSI0E(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSI0E( 4.500) = 0.194
```

BSI1E/DBSI1E (Single/Double precision)

Evaluate the exponentially scaled modified Bessel function of the first kind of order one.

Usage

`BSI1E(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BSI1E — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because the absolute value of `X` is too small.

Algorithm

Function `BSI1E` computes $e^{-|x|} I_1(x)$. For the definition of the Bessel function $I_1(x)$, see `BSI1` (page 91). The function `BSI1E` underflows if $|x|/2$ underflows.

Example

In this example, `BSI1E`(4.5) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BSI1E, VALUE, X
      EXTERNAL   BSI1E, UMACH
C                                 Compute
      X     = 4.5
      VALUE = BSI1E(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSI1E(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSI1E( 4.500) = 0.171
```

BSK0E/DBSK0E (Single/Double precision)

Evaluate the exponentially scaled modified Bessel function of the third kind of order zero.

Usage

`BSK0E(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BSK0E — Function value. (Output)

Algorithm

Function BSK0E computes $e^x K_0(x)$. For the definition of the Bessel function $K_0(x)$, see BSK0 (page 92). The argument must be greater than zero for the result to be defined.

Example

In this example, BSK0E(0.5) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BSK0E, VALUE, X
      EXTERNAL   BSK0E, UMACH
C                                 Compute
      X     = 0.5
      VALUE = BSK0E(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSK0E(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSK0E( 0.500) = 1.524
```

BSK1E/DBSK1E (Single/Double precision)

Evaluate the exponentially scaled modified Bessel function of the third kind of order one.

Usage

BSK1E(X)

Arguments

X — Argument for which the function value is desired. (Input)

BSK1E — Function value. (Output)

Algorithm

Function BSK1E computes $e^x K_1(x)$. For the definition of the Bessel function $K_1(x)$, see BSK1 (page 93). The answer BSK1E = $e^x K_1(x) \approx 1/x$ overflows if x is too close to zero.

Example

In this example, BSK1E(0.5) is computed and printed.

```
C                                 Declare variables
```

```
      INTEGER    NOUT
      REAL       BSK1E, VALUE, X
      EXTERNAL   BSK1E, UMACH
C                                  Compute
      X     = 0.5
      VALUE = BSK1E(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSK1E(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSK1E( 0.500) = 2.731
```

BSJNS/DBSJNS (Single/Double precision)

Evaluate a sequence of Bessel functions of the first kind with integer order and real arguments.

Usage

`CALL BSJNS (X, N, BS)`

Arguments

X — Argument for which the sequence of Bessel functions is to be evaluated. (Input)
Its absolute value must be less than 10^5.

N — Number of elements in the sequence. (Input)
It must be a positive integer.

BS — Vector of length `N` containing the values of the function through the series. (Output)
`BS(I)` contains the value of the Bessel function of order `I` − 1 at x for `I` = 1 to `N`.

Algorithm

The Bessel function $J_n(x)$ is defined to be

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\,d\theta$$

The algorithm is based on a code due to Sookne (1973b) that uses backward recursion with strict error control.

Example

In this example, $J_n(10.0)$, $n = 0, \ldots, 9$ is computed and printed.

```
C                                  Declare variables
```

```
      INTEGER    N
      PARAMETER  (N=10)
C
      INTEGER    K, NOUT
      REAL       BS(N), X
      EXTERNAL   BSJNS, UMACH
C                                 Compute
      X = 10.0
      CALL BSJNS (X, N, BS)
C                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, X, BS(K)
   10 CONTINUE
99999 FORMAT (' J sub ', I2, ' (', F6.3, ') = ', F6.3)
      END
```

Output

```
J sub  0 (10.000) = -0.246
J sub  1 (10.000) =  0.043
J sub  2 (10.000) =  0.255
J sub  3 (10.000) =  0.058
J sub  4 (10.000) = -0.220
J sub  5 (10.000) = -0.234
J sub  6 (10.000) = -0.014
J sub  7 (10.000) =  0.217
J sub  8 (10.000) =  0.318
J sub  9 (10.000) =  0.292
```

CBJNS/DCBJNS (Single/Double precision)

Evaluate a sequence of Bessel functions of the first kind with integer order and complex arguments.

Usage

```
CALL CBJNS (Z, N, CBS)
```

Arguments

Z — Complex argument for which the sequence of Bessel functions is to be evaluated. (Input)
It must be less than 10^4 in absolute value.

N — Number of elements in the sequence. (Input)
It must be positive.

CBS — Vector of length `N` containing the values of the function through the series. (Output)
`CBS(I)` contains the value of the Bessel function of order `I` − 1 at z for `I` = 1 to `N`.

Algorithm

The complex Bessel function $J_n(z)$ is defined to be

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta - n\theta)\,d\theta$$

This code is based on the work of Sookne (1973a) and Olver and Sookne (1972). It uses backward recursion with strict error control.

Example

In this example, $J_n(10 + 10i)$, $n = 0, \ldots, 10$ is computed and printed.

```
C                                  Declare variables
      INTEGER    N
      PARAMETER  (N=11)
C
      INTEGER    K, NOUT
      COMPLEX    CBS(N), Z
      EXTERNAL   CBJNS, UMACH
C                                  Compute
      Z = (10.0, 10.0)
      CALL CBJNS (Z, N, CBS)
C                                  Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' J sub ', I2, ' ((', F6.3, ',', F6.3,
     &       ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
J sub  0 ((10.000,10.000)) = (-2314.975,  411.563)
J sub  1 ((10.000,10.000)) = ( -460.681,-2246.627)
J sub  2 ((10.000,10.000)) = ( 2044.245, -590.157)
J sub  3 ((10.000,10.000)) = (  751.498, 1719.746)
J sub  4 ((10.000,10.000)) = (-1302.871,  880.632)
J sub  5 ((10.000,10.000)) = ( -920.394, -846.345)
J sub  6 ((10.000,10.000)) = (  419.501, -843.607)
J sub  7 ((10.000,10.000)) = (  665.930,   88.480)
J sub  8 ((10.000,10.000)) = (  108.586,  439.392)
J sub  9 ((10.000,10.000)) = ( -227.548,  176.165)
J sub 10 ((10.000,10.000)) = ( -154.831,  -76.050)
```

BSINS/DBSINS (Single/Double precision)

Evaluate a sequence of modified Bessel functions of the first kind with integer order and real arguments.

Usage

```
CALL BSINS (X, N, BSI)
```

Arguments

X — Real argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

BSI — Vector of length `N` containing the values of the function through the series. (Output)
`BSI(I)` contains the value of the Bessel function of order `I` − 1 at x for `I` = 1 to `N`.

Algorithm

The Bessel function $I_n(x)$ is defined to be

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} \exp(x\cos\theta)\cos(n\theta)\,d\theta$$

The input x must satisfy $|x| \le \log(b)$ where b = `AMACH`(2) is the largest representable floating-point number.

The algorithm is based on a code due to Sookne (1973b), which uses backward recursion.

Example

In this example, $I_n(10.0)$, $n = 0, \ldots, 10$ is computed and printed.

```
C                                  Declare variables
      INTEGER    N
      PARAMETER  (N=11)
C
      INTEGER    K, NOUT
      REAL       BSI(N), X
      EXTERNAL   BSINS, UMACH
C                                  Compute
      X = 10.0
      CALL BSINS (X, N, BSI)
C                                  Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, X, BSI(K)
   10 CONTINUE
99999 FORMAT (' I sub ', I2, ' (', F6.3, ') = ', F9.3)
      END
```

Output

```
I sub  0 (10.000) =  2815.716
I sub  1 (10.000) =  2670.988
I sub  2 (10.000) =  2281.519
I sub  3 (10.000) =  1758.381
I sub  4 (10.000) =  1226.490
I sub  5 (10.000) =   777.188
I sub  6 (10.000) =   449.302
```

```
I sub  7 (10.000) =   238.026
I sub  8 (10.000) =   116.066
I sub  9 (10.000) =    52.319
I sub 10 (10.000) =    21.892
```

CBINS/DCBINS (Single/Double precision)

Evaluate a sequence of modified Bessel functions of the first kind with integer order and complex arguments.

Usage

`CALL CBINS (Z, N, CBS)`

Arguments

Z — Complex argument for which the sequence of Bessel functions is to be evaluated. (Input)
It must be less than 10^4 in absolute value.

N — Number of elements in the sequence. (Input)
It must be positive.

CBS — Vector of length `N` containing the values of the function through the series. (Output)
`CBS(I)` contains the value of the Bessel function of order `I` − 1 at z for `I` = 1 to `N`.

Algorithm

The complex Bessel function $I_n(z)$ is defined to be

$$I_n(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta - n\theta)\,d\theta$$

This code is based on the work of Sookne (1973a) and Olver and Sookne (1972). It uses backward recursion with strict error control.

Example

In this example, $I_n(10 + 10i)$, $n = 0, \ldots, 10$ is computed and printed.

```
C                                Declare variables
      INTEGER    N
      PARAMETER  (N=11)
C
      INTEGER    K, NOUT
      COMPLEX    CBS(N), Z
      EXTERNAL   CBINS, UMACH
C                                Compute
      Z = (10.0, 10.0)
      CALL CBINS (Z, N, CBS)
C                                Print the results
```

```
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' I sub ', I2, ' ((', F6.3, ',', F6.3,
     &       ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
I sub  0 ((10.000,10.000)) = (-2314.975, -411.563)
I sub  1 ((10.000,10.000)) = (-2246.627, -460.681)
I sub  2 ((10.000,10.000)) = (-2044.245, -590.157)
I sub  3 ((10.000,10.000)) = (-1719.746, -751.498)
I sub  4 ((10.000,10.000)) = (-1302.871, -880.632)
I sub  5 ((10.000,10.000)) = ( -846.345, -920.394)
I sub  6 ((10.000,10.000)) = ( -419.501, -843.607)
I sub  7 ((10.000,10.000)) = (  -88.480, -665.930)
I sub  8 ((10.000,10.000)) = (  108.586, -439.392)
I sub  9 ((10.000,10.000)) = (  176.165, -227.548)
I sub 10 ((10.000,10.000)) = (  154.831,  -76.050)
```

BSJS/DBSJS (Single/Double precision)

Evaluate a sequence of Bessel functions of the first kind with real order and real positive arguments.

Usage

`CALL BSJS (XNU, X, N, BS)`

Arguments

XNU — Real argument which is the lowest order desired. (Input)
It must be at least zero and less than one.

X — Real argument for which the sequence of Bessel functions is to be evaluated. (Input)
It must be nonnegative.

N — Number of elements in the sequence. (Input)

BS — Vector of length `N` containing the values of the function through the series. (Output)
`BS(I)` contains the value of the Bessel function of order `XNU` + `I` − 1 at x for `I` = 1 to `N`.

Comments

Automatic workspace usage is

`BSJS` 2 ∗ `N` units, or
`DBSJS` 4 ∗ `N` units.

Workspace may be explicitly provided, if desired, by use of B2JS/DB2JS. The reference is

CALL B2JS (XNU, X, N, BS, WK)

The additional argument is

WK — work array of length 2 ∗ N.

Algorithm

The Bessel function $J_\nu(x)$ is defined to be

$$J_\nu(x) = \frac{(x/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_0^\pi \cos(x \cos\theta) \sin^{2\nu}\theta \, d\theta$$

This code is based on the work of Gautschi (1964) and Skovgaard (1975). It uses backward recursion.

Example

In this example, $J_\nu(2.4048256)$, $\nu = 0, \ldots, 10$ is computed and printed.

```
C                                 Declare variables
      INTEGER    N
      PARAMETER  (N=11)
C
      INTEGER    K, NOUT
      REAL       BS(N), X, XNU
      EXTERNAL   BSJS, UMACH
C                                 Compute
      XNU = 0.0
      X   = 2.4048256
      CALL BSJS (XNU, X, N, BS)
C                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, X, BS(K)
   10 CONTINUE
99999 FORMAT (' J sub ', F6.3, ' (', F6.3, ') = ', F10.3)
      END
```

Output

```
J sub  0.000 ( 2.405) =      0.000
J sub  1.000 ( 2.405) =      0.519
J sub  2.000 ( 2.405) =      0.432
J sub  3.000 ( 2.405) =      0.199
J sub  4.000 ( 2.405) =      0.065
J sub  5.000 ( 2.405) =      0.016
J sub  6.000 ( 2.405) =      0.003
J sub  7.000 ( 2.405) =      0.001
J sub  8.000 ( 2.405) =      0.000
J sub  9.000 ( 2.405) =      0.000
J sub 10.000 ( 2.405) =      0.000
```

BSYS/DBSYS (Single/Double precision)

Evaluate a sequence of Bessel functions of the second kind with real nonnegative order and real positive arguments.

Usage

```
CALL BSYS (XNU, X, N, BSY)
```

Arguments

XNU — Real argument which is the lowest order desired. (Input)
It must be at least zero and less than one.

X — Real positive argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

BSY — Vector of length `N` containing the values of the function through the series. (Output)
`BSY(I)` contains the value of the Bessel function of order `I` − 1 + `XNU` at x for `I` = 1 to `N`.

Algorithm

The Bessel function $Y_\nu(x)$ is defined to be

$$Y_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - \nu\theta)\,d\theta - \frac{1}{\pi}\int_0^\infty \left[e^{\nu t} + e^{-\nu t}\cos(\nu\pi)\right]e^{-x\sinh t}\,dt$$

The variable ν must satisfy $0 \le \nu < 1$. If this condition is not met, then $\mathtt{BS}_i$ is set to $-b$. In addition, x must be in $[x_m, x_M]$ where $x_m = 6(16^{-32})$ and $x_M = 16^9$. If $x < x_m$, then $-b$ (b = `AMACH`(2), the largest representable number) is returned; and if $x > x_M$, then zero is returned.

The algorithm is based on work of Cody and others, (see Cody et al. 1976; Cody 1969; *NATS FUNPACK* 1976). It uses a special series expansion for small arguments. For moderate arguments, an analytic continuation in the argument based on Taylor series with special rational minimax approximations providing starting values is employed. An asymptotic expansion is used for large arguments.

Example

In this example, $Y_{0.015625 + \nu - 1}(0.0078125)$, $\nu = 1, 2, 3$ is computed and printed.

```
C                                 Declare variables
```

```
      INTEGER    N
      PARAMETER  (N=3)
C
      INTEGER    K, NOUT
      REAL       BSY(N), X, XNU
      EXTERNAL   BSYS, UMACH
C                                 Compute
      XNU = 0.015625
      X   = 0.0078125
      CALL BSYS (XNU, X, N, BSY)
C                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, X, BSY(K)
   10 CONTINUE
99999 FORMAT (' Y sub ', F6.3, ' (', F6.3, ') = ', F10.3)
      END
```

Output

```
Y sub  0.016 ( 0.008) =     -3.189
Y sub  1.016 ( 0.008) =    -88.096
Y sub  2.016 ( 0.008) = -22901.732
```

BSIS/DBSIS (Single/Double precision)

Evaluate a sequence of modified Bessel functions of the first kind with real order and real positive arguments.

Usage

`CALL BSIS (XNU, X, N, BSI)`

Arguments

XNU — Real argument which is the lowest order desired. (Input)
It must be greater than or equal to zero and less than one.

X — Real argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

BSI — Vector of length `N` containing the values of the function through the series. (Output)
`BSI(I)` contains the value of the Bessel function of order `I` − 1 + `XNU` at x for `I` = 1 to N.

Algorithm

The Bessel function $I_\nu(x)$ is defined to be

$$I_\nu(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-x\cosh t - \nu t}\,dt$$

The input x must be nonnegative and less than or equal to $\log(b)$ (b = AMACH(2), the largest representable number). The argument ν = XNU must satisfy $0 \le \nu \le 1$.

Function BSIS is based on a code due to Cody (1983), which uses backward recursion.

Example

In this example, $I_{\nu-1}(10.0)$, $\nu = 1, \ldots, 10$ is computed and printed.

```
C                                 Declare variables
      INTEGER    N
      PARAMETER  (N=10)
C
      INTEGER    K, NOUT
      REAL       BSI(N), X, XNU
      EXTERNAL   BSIS, UMACH
C                                 Compute
      XNU = 0.0
      X   = 10.0
      CALL BSIS (XNU, X, N, BSI)
C                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, X, BSI(K)
   10 CONTINUE
99999 FORMAT (' I sub ', F6.3, ' (', F6.3, ') = ', F10.3)
      END
```

Output

```
I sub  0.000 (10.000) =   2815.717
I sub  1.000 (10.000) =   2670.988
I sub  2.000 (10.000) =   2281.519
I sub  3.000 (10.000) =   1758.381
I sub  4.000 (10.000) =   1226.491
I sub  5.000 (10.000) =    777.188
I sub  6.000 (10.000) =    449.302
I sub  7.000 (10.000) =    238.026
I sub  8.000 (10.000) =    116.066
I sub  9.000 (10.000) =     52.319
```

BSIES/DBSIES (Single/Double precision)

Evaluate a sequence of exponentially scaled modified Bessel functions of the first kind with nonnegative real order and real positive arguments.

Usage

```
CALL BSIES (XNU, X, N, BSI)
```

Arguments

XNU — Real argument which is the lowest order desired. (Input)
It must be at least zero and less than one.

X — Real positive argument for which the sequence of Bessel functions is to be evaluated. (Input)
It must be nonnegative and less than 10^4.

N — Number of elements in the sequence. (Input)

BSI — Vector of length `N` containing the values of the function through the series. (Output)
`BSI(I)` contains the value of the Bessel function of order `I` − 1 + `XNU` at x for `I` = 1 to `N` multiplied by exp(−`X`).

Algorithm

Function `BSIES` evaluates $e^{-x} I_{\nu + k - 1}(x)$, for $k = 1, \ldots, n$. For the definition of $I_\nu(x)$, see `BSIS` (page 106). The algorithm is based on a code due to Cody (1983), which uses backward recursion.

Example

In this example, $I_{\nu - 1}(10.0)$, $\nu = 1, \ldots, 10$ is computed and printed.

```
C                                 Declare variables
      INTEGER    N
      PARAMETER  (N=10)
C
      INTEGER    K, NOUT
      REAL       BSI(N), X, XNU
      EXTERNAL   BSIES, UMACH
C                                 Compute
      XNU = 0.0
      X   = 10.0
      CALL BSIES (XNU, X, N, BSI)
C                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) X, XNU+K-1, X, BSI(K)
   10 CONTINUE
99999 FORMAT (' exp(-', F6.3, ') * I sub ', F6.3,
     &       ' (', F6.3, ') = ', F6.3)
      END
```

Output

```
exp(-10.000) * I sub  0.000 (10.000) =  0.128
exp(-10.000) * I sub  1.000 (10.000) =  0.121
exp(-10.000) * I sub  2.000 (10.000) =  0.104
exp(-10.000) * I sub  3.000 (10.000) =  0.080
exp(-10.000) * I sub  4.000 (10.000) =  0.056
exp(-10.000) * I sub  5.000 (10.000) =  0.035
exp(-10.000) * I sub  6.000 (10.000) =  0.020
```

```
exp(-10.000) * I sub  7.000 (10.000) =  0.011
exp(-10.000) * I sub  8.000 (10.000) =  0.005
exp(-10.000) * I sub  9.000 (10.000) =  0.002
```

BSKS/DBSKS (Single/Double precision)

Evaluate a sequence of modified Bessel functions of the third kind of fractional order.

Usage

CALL BSKS (XNU, X, NIN, BK)

Arguments

XNU — Fractional order of the function. (Input)
XNU must be less than one in absolute value.

X — Argument for which the sequence of Bessel functions is to be evaluated. (Input)

NIN — Number of elements in the sequence. (Input)

BK — Vector of length NIN containing the values of the function through the series. (Output)

Comments

1. If NIN is positive, BK(1) contains the value of the function of order XNU, BK(2) contains the value of the function of order XNU + 1, … and BK(NIN) contains the value of the function of order XNU + NIN – 1.

2. If NIN is negative, BK(1) contains the value of the function of order XNU, BK(2) contains the value of the function of order XNU – 1, … and BK(ABS(NIN)) contains the value of the function of order XNU + NIN + 1.

Algorithm

The Bessel function $K_\nu(x)$ is defined to be

$$K_\nu(x) = \frac{\pi}{2} e^{\nu\pi i/2}\left[i J_\nu(ix) - Y_\nu(ix)\right] \quad \text{for } -\pi < \arg x \le \frac{\pi}{2}$$

Currently, ν is restricted to be less than one in absolute value. A total of $|n|$ values is stored in the array BK. For positive n, BK(1) = $K_\nu(x)$, BK(2) = $K_{\nu+1}(x)$, …, BK(n) = $K_{\nu+n-1}(x)$. For negative n, BK(1) = $K_\nu(x)$, BK(2) = $K_{\nu-1}(x)$, …, BK($|n|$) = $K_{\nu+n+1}$.

BSKS is based on the work of Cody (1983).

Example

In this example, $K_{\nu-1}(10.0)$, $\nu = 1, \ldots, 10$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NIN
      PARAMETER  (NIN=10)
C
      INTEGER    K, NOUT
      REAL       BS(NIN), X, XNU
      EXTERNAL   BSKS, UMACH
C                                 Compute
      XNU = 0.0
      X   = 10.0
      CALL BSKS (XNU, X, NIN, BS)
C                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, NIN
         WRITE (NOUT,99999) XNU+K-1, X, BS(K)
   10 CONTINUE
99999 FORMAT (' K sub ', F6.3, ' (', F6.3, ') = ', E10.3)
      END
```

Output

```
K sub  0.000 (10.000) =  0.178E-04
K sub  1.000 (10.000) =  0.186E-04
K sub  2.000 (10.000) =  0.215E-04
K sub  3.000 (10.000) =  0.273E-04
K sub  4.000 (10.000) =  0.379E-04
K sub  5.000 (10.000) =  0.575E-04
K sub  6.000 (10.000) =  0.954E-04
K sub  7.000 (10.000) =  0.172E-03
K sub  8.000 (10.000) =  0.336E-03
K sub  9.000 (10.000) =  0.710E-03
```

BSKES/DBSKES (Single/Double precision)

Evaluate a sequence of exponentially scaled modified Bessel functions of the third kind of fractional order.

Usage

CALL BSKES (XNU, X, NIN, BKE)

Arguments

XNU — Fractional order of the function. (Input)
XNU must be less than 1.0 in absolute value.

X — Argument for which the sequence of Bessel functions is to be evaluated. (Input)

NIN — Number of elements in the sequence. (Input)

BKE — Vector of length `NIN` containing the values of the function through the series. (Output)

Comments

1. If `NIN` is positive, `BKE`(1) contains `EXP(X)` times the value of the function of order `XNU`, `BKE`(2) contains `EXP(X)` times the value of the function of order `XNU` + 1, …, and `BKE(NIN)` contains `EXP(X)` times the value of the function of order `XNU` + `NIN` – 1.
2. If `NIN` is negative, `BKE`(1) contains `EXP(X)` times the value of the function of order `XNU`, `BKE`(2) contains `EXP(X)` times the value of the function of order `XNU` – 1, …, and `BKE(ABS(NIN))` contains `EXP(X)` times the value of the function of order `XNU` + `NIN` + 1.

Algorithm

Function `BSKES` evaluates $e^x K_{\nu+k-1}(x)$, for $k = 1, \ldots, n$. For the definition of $K_\nu(x)$, see `BSKS` (page 109).

Currently, ν is restricted to be less than 1 in absolute value. A total of $|n|$ values is stored in the array `BKE`. For n positive, `BKE`(1) contains $e^x K\nu(x)$, `BKE`(2) contains $e^x K_{\nu+1}(x)$, …, and `BKE(N)` contains $e^x K_{\nu+n-1}(x)$. For n negative, `BKE`(1) contains $e^x K_\nu(x)$, `BKE`(2) contains $e^x K_{\nu-1}(x)$, …, and `BKE`($|n|$) contains $e^x K_{\nu+n+1}(x)$. This routine is particularly useful for calculating sequences for large x provided $n \le x$. (Overflow becomes a problem if $n \ll x$.) n must not be zero, and x must not be greater than zero. Moreover, $|\nu|$ must be less than 1. Also, when $|n|$ is large compared with x, $|\nu + n|$ must not be so large that $e^x K_{\nu+n}(x) \approx e^x \Gamma(|\nu + n|)/[2(x/2)^{|\nu+n|}]$ overflows.

`BSKES` is based on the work of Cody (1983).

Example

In this example, $K_{\nu-1/2}(2.0)$, $\nu = 1, \ldots, 6$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NIN
      PARAMETER  (NIN=6)
C
      INTEGER    K, NOUT
      REAL       BKE(NIN), X, XNU
      EXTERNAL   BSKES, UMACH
C                                 Compute
      XNU = 0.5
      X   = 2.0
      CALL BSKES (XNU, X, NIN, BKE)
C                                 Print the results
      CALL UMACH (2, NOUT)
```

```
      DO 10  K=1, NIN
         WRITE (NOUT,99999) X, XNU+K-1, X, BKE(K)
   10 CONTINUE
99999 FORMAT (' exp(', F6.3, ') * K sub ', F6.3,
     &      ' (', F6.3, ') = ', F8.3)
      END
```

Output

```
exp( 2.000) * K sub  0.500 ( 2.000) =    0.886
exp( 2.000) * K sub  1.500 ( 2.000) =    1.329
exp( 2.000) * K sub  2.500 ( 2.000) =    2.880
exp( 2.000) * K sub  3.500 ( 2.000) =    8.530
exp( 2.000) * K sub  4.500 ( 2.000) =   32.735
exp( 2.000) * K sub  5.500 ( 2.000) =  155.837
```

CBJS/DCBJS (Single/Double precision)

Evaluate a sequence of Bessel functions of the first kind with real order and complex arguments.

Usage

`CALL CBJS (XNU, Z, N, CBS)`

Arguments

XNU — Real argument which is the lowest order desired. (Input)
`XNU` must be greater than −1/2.

Z — Complex argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

CBS — Vector of length `N` containing the values of the function through the series. (Output)
`CBS(I)` contains the value of the Bessel function of order `XNU` + `I` − 1 at `Z` for `I` = 1 to `N`.

Comments

Informational errors

Type	Code	
3	1	One of the continued fractions failed.
4	2	Only the first several entries in `CBS` are valid.

Algorithm

The Bessel function $J_\nu(z)$ is defined to be

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta - \nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{z\sinh t - \nu t}\,dt$$

$$\text{for } |\arg z| < \frac{\pi}{2}$$

This code is based on the code `BESSCC` of Barnett (1981) and Thompson and Barnett (1987).

This code computes $J_\nu(z)$ from the modified Bessel function $I_\nu(z)$ (see page 116), using the following relation, with $\rho = e^{i\pi/2}$:

$$Y_\nu(z) = \begin{cases} \rho I_\nu(z/\rho) & \text{for } -\pi/2 < \arg z \le \pi \\ \rho^3 I_\nu(\rho^3 z) & \text{for } -\pi < \arg z \le \pi/2 \end{cases}$$

Example

In this example, $J_{0.3+\nu-1}(1.2 + 0.5i)$, $\nu = 1, \ldots, 4$ is computed and printed.

```
C                                 Declare variables
      INTEGER    N
      PARAMETER  (N=4)
C
      INTEGER    K, NOUT
      REAL       XNU
      COMPLEX    CBS(N), Z
      EXTERNAL   CBJS, UMACH
C                                 Compute
      XNU = 0.3
      Z   = (1.2, 0.5)
      CALL CBJS (XNU, Z, N, CBS)
C                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' J sub ', F6.3, ' ((', F6.3, ',', F6.3,
     &       ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
J sub  0.300 (( 1.200, 0.500)) = (    0.774,   -0.107)
J sub  1.300 (( 1.200, 0.500)) = (    0.400,    0.159)
J sub  2.300 (( 1.200, 0.500)) = (    0.087,    0.092)
J sub  3.300 (( 1.200, 0.500)) = (    0.008,    0.024)
```

CBYS/DCBYS (Single/Double precision)

Evaluate a sequence of Bessel functions of the second kind with real order and complex arguments.

Usage

```
CALL CBYS (XNU, Z, N, CBS)
```

Arguments

XNU — Real argument which is the lowest order desired. (Input)
`XNU` must be greater than −1/2.

Z — Complex argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

CBS — Vector of length `N` containing the values of the function through the series. (Output)
`CBS(I)` contains the value of the Bessel function of order `XNU` + `I` − 1 at `Z` for `I` = 1 to `N`.

Comments

1. Automatic workspace usage is

   `CBYS` 2 ∗ `N` units, or
   `DCBYS` 4 ∗ `N` units.

   Workspace may be explicitly provided, if desired, by use of `C2YS`/`DC2YS`. The reference is

   ```
   CALL C2YS (XNU, Z, N, CBS, FK)
   ```

   The additional argument is

   FK — complex work vector of length `N`.

2. Informational errors

Type	Code	
3	1	One of the continued fractions failed.
4	2	Only the first several entries in `CBS` are valid.

Algorithm

The Bessel function $Y_\nu(z)$ is defined to be

$$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin\theta - \nu\theta)d\theta$$

$$-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty \left[e^{\nu t} + e^{-\nu t}\cos(\nu t)\right]e^{z\sinh t}\,dt$$

$$\text{for } |\arg z| < \frac{\pi}{2}$$

This code is based on the code `BESSEC` of Barnett (1981) and Thompson and Barnett (1987).

This code computes $Y_\nu(z)$ from the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ (see `CBIS`, page 115, and `CBKS`, page 117), using the following relation:

$$Y_\nu(z) = e^{(\nu+1)\pi i/2} I_\nu(z) - \frac{2}{\pi} e^{-\nu\pi i/2} K_\nu(z) \quad \text{for } -\pi < \arg z \le \pi/2$$

Example

In this example, $Y_\nu 0.3 + n - 1(1.2 + 0.5i)$, $\nu = 1, \ldots, 4$ is computed and printed.

```
C                                  Declare variables
      INTEGER    N
      PARAMETER  (N=4)
C
      INTEGER    K, NOUT
      REAL       XNU
      COMPLEX    CBS(N), Z
      EXTERNAL   CBYS, UMACH
C                                  Compute
      XNU = 0.3
      Z   = (1.2, 0.5)
      CALL CBYS (XNU, Z, N, CBS)
C                                  Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' Y sub ', F6.3, ' ((', F6.3, ',', F6.3,
     &     ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
Y sub  0.300 (( 1.200, 0.500)) = (   -0.013,    0.380)
Y sub  1.300 (( 1.200, 0.500)) = (   -0.716,    0.338)
Y sub  2.300 (( 1.200, 0.500)) = (   -1.048,    0.795)
Y sub  3.300 (( 1.200, 0.500)) = (   -1.625,    3.684)
```

CBIS/DCBIS (Single/Double precision)

Evaluate a sequence of modified Bessel functions of the first kind with real order and complex arguments.

Usage

`CALL CBIS (XNU, Z, N, CBS)`

Arguments

XNU — Real argument which is the lowest order desired. (Input)
`XNU` must be greater than –1/2.

Z — Complex argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

CBS — Vector of length `N` containing the values of the function through the series. (Output)
`CBS(I)` contains the value of the Bessel function of order `XNU` + `I` – 1 at `Z` for `I` = 1 to `N`.

Comments

Informational errors

Type	Code	
3	1	One of the continued fractions failed.
4	2	Only the first several entries in `CBS` are valid.

Algorithm

The modified Bessel function $I_\nu(z)$ is defined to be

$$I_v(z) = e^{-v\pi i/2} J_v(ze^{\pi i/2}) \quad \text{for} -\pi < \arg z \le \frac{\pi}{2}$$

where the Bessel function $J_\nu(z)$ is defined in `BSJS` (page 103).

This code is based on the code `BESSCC` of Barnett (1981) and Thompson and Barnett (1987).

For large arguments, z, Temme's (1975) algorithm is used to find $I_\nu(z)$. The $I_\nu(z)$ values are recurred upward (if this is stable). This involves evaluating a continued fraction. If this evaluation fails to converge, the answer may not be accurate. For moderate and small arguments, Miller's method is used.

Example

In this example, $I_{0.3 + \nu - 1}(1.2 + 0.5i)$, $\nu = 1, \ldots, 4$ is computed and printed.

```
C                                 Declare variables
      INTEGER    N
      PARAMETER  (N=4)
C
      INTEGER    K, NOUT
      REAL       XNU
      COMPLEX    CBS(N), Z
      EXTERNAL   CBIS, UMACH
C                                 Compute
      XNU = 0.3
      Z   = (1.2, 0.5)
      CALL CBIS (XNU, Z, N, CBS)
C                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' I sub ', F6.3, ' ((', F6.3, ',', F6.3,
     &       ')) = (', F9.3, ',', F9.3, ')')
```

```
      END
```

Output

```
I sub  0.300 (( 1.200, 0.500)) = (    1.163,    0.396)
I sub  1.300 (( 1.200, 0.500)) = (    0.447,    0.332)
I sub  2.300 (( 1.200, 0.500)) = (    0.082,    0.127)
I sub  3.300 (( 1.200, 0.500)) = (    0.006,    0.029)
```

CBKS/DCBKS (Single/Double precision)

Evaluate a sequence of modified Bessel functions of the second kind with real order and complex arguments.

Usage

`CALL CBKS (XNU, Z, N, CBS)`

Arguments

XNU — Real argument which is the lowest order desired. (Input)
`XNU` must be greater than −1/2.

Z — Complex argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

CBS — Vector of length `N` containing the values of the function through the series. (Output)
`CBS(I)` contains the value of the Bessel function of order `XNU` + `I` − 1 at `Z` for `I` = 1 to `N`.

Comments

1. Automatic workspace usage is

   `CBKS` 2 ∗ `N` units, or
   `DCBKS` 4 ∗ `N` units.

   Workspace may be explicitly provided, if desired, by use of `C2KS`/`DC2KS`. The reference is

   `CALL C2KS (XNU, Z, N, CBS, FK)`

   The additional argument is

   FK — Complex work vector of length `N`.

2. Informational errors

Type	Code	
3	1	One of the continued fractions failed.
4	2	Only the first several entries in `CBS` are valid.

Algorithm

The Bessel function $K_\nu(z)$ is defined to be

$$K_\nu(z) = \frac{\pi}{2} e^{\nu\pi i/2}\left[iJ_\nu(iz) - Y_\nu(iz)\right] \quad \text{for} -\pi < \arg z \le \frac{\pi}{2}$$

where the Bessel function $J_\nu(z)$ is defined in `CBJS` (page 112) and $Y_\nu(z)$ is defined in `CBYS` (page 113).

This code is based on the code `BESSCC` of Barnett (1981) and Thompson and Barnett (1987).

For moderate or large arguments, z, Temme's (1975) algorithm is used to find $K_\nu(z)$. This involves evaluating a continued fraction. If this evaluation fails to converge, the answer may not be accurate. For small z, a Neumann series is used to compute $K_\nu(z)$. Upward recurrence of the $K_\nu(z)$ is always stable.

Example

In this example, $K_{0.3 + \nu - 1}(1.2 + 0.5i)$, $\nu = 1, \ldots, 4$ is computed and printed.

```
C                                  Declare variables
      INTEGER    N
      PARAMETER  (N=4)
C
      INTEGER    K, NOUT
      REAL       XNU
      COMPLEX    CBS(N), Z
      EXTERNAL   CBKS, UMACH
C                                  Compute
      XNU = 0.3
      Z   = (1.2, 0.5)
      CALL CBKS (XNU, Z, N, CBS)
C                                  Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' K sub ', F6.3, ' ((', F6.3, ',', F6.3,
     &       ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
K sub  0.300 (( 1.200, 0.500)) = (    0.246,   -0.200)
K sub  1.300 (( 1.200, 0.500)) = (    0.336,   -0.362)
K sub  2.300 (( 1.200, 0.500)) = (    0.587,   -1.126)
K sub  3.300 (( 1.200, 0.500)) = (    0.719,   -4.839)
```

Chapter 7: Kelvin Functions

Routines

Evaluate $\mathrm{ber}_0(x)$ BER0 121
Evaluate $\mathrm{bei}_0(x)$ BEI0 122
Evaluate $\mathrm{ker}_0(x)$ AKER0 123
Evaluate $\mathrm{kei}_0(x)$ AKEI0 124
Evaluate $\mathrm{ber}'_0(x)$ BERP0 124
Evaluate $\mathrm{bei}_0(x)$ BEIP0 125
Evaluate $\mathrm{ker}'_0(x)$ AKERP0 126
Evaluate $\mathrm{kei}'_0(x)$ AKEIP0 127
Evaluate $\mathrm{ber}_1(x)$ BER1 128
Evaluate $\mathrm{bei}_1(x)$ BEI1 129
Evaluate $\mathrm{ker}_1(x)$ AKER1 130
Evaluate $\mathrm{kei}_1(x)$ AKEI1 130

Usage Notes

The notation used in this chapter follows that of Abramowitz and Stegun (1964). The Kelvin functions are related to the Bessel functions by the following relations.

$$\mathrm{ber}_v x + i\,\mathrm{bei}_v x = J_v(xe^{3\pi i/4})$$

$$\mathrm{ker}_v\, x + i\,\mathrm{kei}_v x = e^{-\pi i/2} K_v(xe^{\pi i/4})$$

The derivatives of the Kelvin functions are related to the values of the Kelvin functions by the following:

$$\sqrt{2}\mathrm{ber}'_0 x = \mathrm{ber}_1 x + \mathrm{bei}_1 x$$

$$\sqrt{2}\mathrm{bei}'_0 x = -\mathrm{ber}_1 x + \mathrm{bei}_1 x$$

$$\sqrt{2}\mathrm{ker}'_0 x = \mathrm{ker}_1 x + \mathrm{kei}_1 x$$

$$\sqrt{2}\,\mathrm{kei}_0'\,x = -\mathrm{ker}_1 x + \mathrm{kei}_1 x$$

Plots of $\mathrm{ber}_n(x)$, $\mathrm{bei}_n(x)$, $\mathrm{ker}_n(x)$ and $\mathrm{kei}_n(x)$ for $n = 0, 1$ follow:

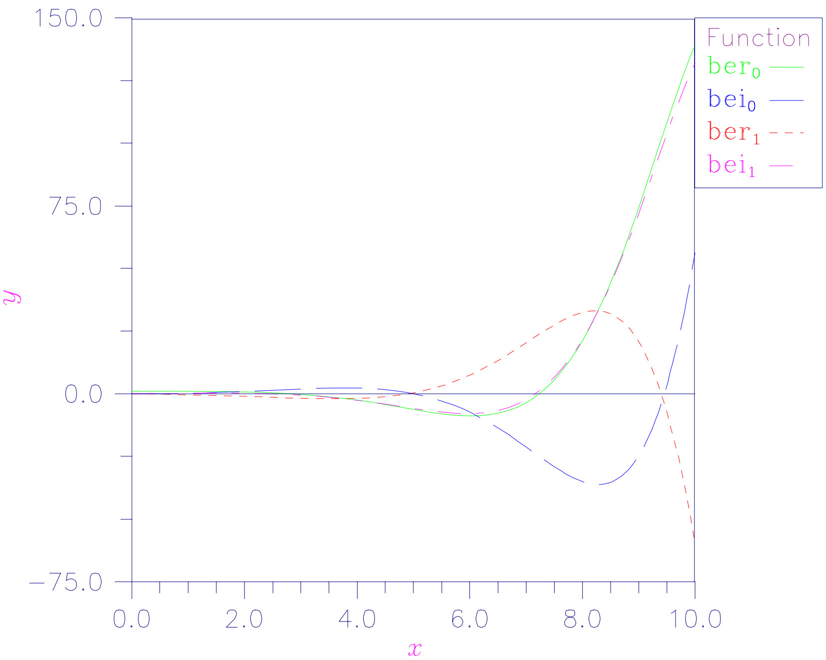

Figure 7-1   Plot of $\mathrm{ber}_n(x)$ and $\mathrm{bei}_n(x)$

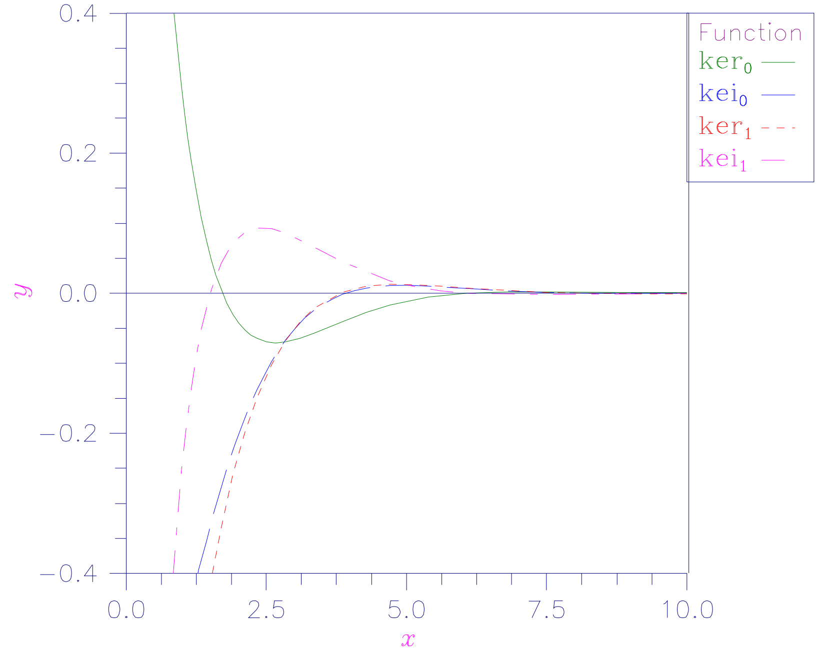

Figure 7-2 Plot of $\mathrm{ker}_n(x)$ and $\mathrm{kei}_n(x)$

BER0/DBER0 (Single/Double precision)

Evaluate the Kelvin function of the first kind, ber, of order zero.

Usage

`BER0(X)`

Arguments

X — Argument for which the function value is desired. (Input)
`ABS(X)` must be less than 119.

BER0 — Function value. (Output)

Algorithm

The Kelvin function $\mathrm{ber}_0(x)$ is defined to be $\Re J_0(xe^{3\pi i/4})$. The Bessel function $J_0(x)$ is defined in `BSJ0` (page 84). Function `BER0` is based on the work of Burgoyne (1963).

Example

In this example, $\mathrm{ber}_0(0.4)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BER0, VALUE, X
      EXTERNAL   BER0, UMACH
C                                 Compute
      X     = 0.4
      VALUE = BER0(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BER0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BER0( 0.400) = 1.000
```

BEI0/DBEI0 (Single/Double precision)

Evaluate the Kelvin function of the first kind, bei, of order zero.

Usage

`BEI0(X)`

Arguments

X — Argument for which the function value is desired. (Input)
`ABS(X)` must be less than 119.

BEI0 — Function value. (Output)

Algorithm

The Kelvin function $\mathrm{bei}_0(x)$ is defined to be $\Im J_0(xe^{3\pi i/4})$. The Bessel function $J_0(x)$ is defined in `BSJ0` (page 84). Function `BEI0` is based on the work of Burgoyne (1963).

In `BEI0`, x must be less than 119.

Example

In this example, $\mathrm{bei}_0(0.4)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BEI0, VALUE, X
      EXTERNAL   BEI0, UMACH
C                                 Compute
      X     = 0.4
```

```
      VALUE = BEI0(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BEI0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BEI0( 0.400) = 0.040
```

AKER0/DKER0 (Single/Double precision)

Evaluate the Kelvin function of the second kind, ker, of order zero.

Usage

AKER0(X)

Arguments

X — Argument for which the function value is desired. (Input)
It must be nonnegative.

AKER0 — Function value. (Output)

Algorithm

The modified Kelvin function $\mathrm{ker}_0(x)$ is defined to be $\Re K_0(xe^{\pi i/4})$. The Bessel function $K_0(x)$ is defined in BSK0 (page 92). Function AKER0 is based on the work of Burgoyne (1963). If $x < 0$, then NaN (not a number) is returned. If $x \geq 119$, then zero is returned.

Example

In this example, $\mathrm{ker}_0(0.4)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AKER0, VALUE, X
      EXTERNAL   AKER0, UMACH
C                                 Compute
      X     = 0.4
      VALUE = AKER0(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKER0(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKER0( 0.400) = 1.063
```

AKEI0/DKEI0 (Single/Double precision)

Evaluate the Kelvin function of the second kind, kei, of order zero.

Usage

`AKEI0(X)`

Arguments

X — Argument for which the function value is desired. (Input)
It must be nonnegative and less than 119.

AKEI0 — Function value. (Output)

Algorithm

The modified Kelvin function $\mathrm{kei}_0(x)$ is defined to be $\Im K_0(xe^{\pi i/4})$. The Bessel function $K_0(x)$ is defined in `BSK0` (page 92). Function `AKEI0` is based on the work of Burgoyne (1963).

In `AKEI0`, x must satisfy $0 \le x < 119$. If $x < 0$, then NaN (not a number) is returned. If $x \ge 119$, then zero is returned.

Example

In this example, $\mathrm{kei}_0(0.4)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AKEI0, VALUE, X
      EXTERNAL   AKEI0, UMACH
C                                 Compute
      X     = 0.4
      VALUE = AKEI0(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKEI0(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKEI0( 0.400) = -0.704
```

BERP0/DBERP0 (Single/Double precision)

Evaluate the derivative of the Kelvin function of the first kind, ber, of order zero.

Usage

`BERP0(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BERP0 — Function value. (Output)

Algorithm

The function $\mathrm{ber}'_0(x)$ is defined to be

$$\frac{d}{dx}\,\mathrm{ber}_0(x)$$

where $\mathrm{ber}_0(x)$ is a Kelvin function, see `BER0` (page 121). Function `BERP0` is based on the work of Burgoyne (1963).

If |x| > 119, then NaN (not a number) is returned.

Example

In this example, $\mathrm{ber}'_0(0.6)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BERP0, VALUE, X
      EXTERNAL   BERP0, UMACH
C                                 Compute
      X     = 0.6
      VALUE = BERP0(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BERP0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BERP0( 0.600) = -0.013
```

BEIP0/DBEIP0 (Single/Double precision)

Evaluate the derivative of the Kelvin function of the first kind, bei, of order zero.

Usage

`BEIP0(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BEIP0 — Function value.   (Output)

Algorithm

The function $\mathrm{bei}'_0(x)$ is defined to be

$$\frac{d}{dx}\,\mathrm{bei}_0(x)$$

where $\mathrm{bei}_0(x)$ is a Kelvin function, see `BEI0` (page 122). Function `BEIP0` is based on the work of Burgoyne (1963).

If $|x| > 119$, then NaN (not a number) is returned.

Example

In this example, $\mathrm{bei}'_0(0.6)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       BEIP0, VALUE, X
      EXTERNAL   BEIP0, UMACH
C                                  Compute
      X     = 0.6
      VALUE = BEIP0(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BEIP0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BEIP0( 0.600) = 0.300
```

AKERP0/DKERP0 (Single/Double precision)

Evaluate the derivative of the Kelvin function of the second kind, ker, of order zero.

Usage

`AKERP0(X)`

Arguments

X — Argument for which the function value is desired.   (Input)
It must be nonnegative.

AKERP0 — Function value.   (Output)

Algorithm

The function $\mathrm{ker}'_0(x)$ is defined to be

$$\frac{d}{dx}\ \mathrm{ker}_0(x)$$

where $\mathrm{ker}_0(x)$ is a Kelvin function, see `AKER0` (page 123). Function `AKERP0` is based on the work of Burgoyne (1963). If $x < 0$, then NaN (not a number) is returned. If $x > 119$, then zero is returned.

Example

In this example, $\mathrm{ker}'_0(0.6)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       AKERP0, VALUE, X
      EXTERNAL   AKERP0, UMACH
C                                  Compute
      X     = 0.6
      VALUE = AKERP0(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKERP0(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKERP0( 0.600) = -1.457
```

AKEIP0/DKEIP0 (Single/Double precision)

Evaluate the Kelvin function of the second kind, kei, of order zero.

Usage

`AKEIP0(X)`

Arguments

X — Argument for which the function value is desired. (Input)
It must be nonnegative.

AKEIP0 — Function value. (Output)

Algorithm

The function $\mathrm{kei}'_0(x)$ is defined to be

$$\frac{d}{dx}\ \mathrm{kei}_0(x)$$

where $\mathrm{kei}_0(x)$ is a Kelvin function, see `AKEIP0` (page 127). Function `AKEIP0` is based on the work of Burgoyne (1963).

If $x < 0$, then NaN (not a number) is returned. If $x > 119$, then zero is returned.

Example

In this example, $\text{kei}'_0(0.6)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AKEIP0, VALUE, X
      EXTERNAL   AKEIP0, UMACH
C                                 Compute
      X     = 0.6
      VALUE = AKEIP0(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKEIP0(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKEIP0( 0.600) = 0.348
```

BER1/DBER1 (Single/Double precision)

Evaluate the Kelvin function of the first kind, ber, of order one.

Usage

`BER1(X)`

Arguments

X — Argument for which the function value is desired. (Input)

BER1 — Function value. (Output)

Algorithm

The Kelvin function $\text{ber}_1(x)$ is defined to be $\Re J_1(xe^{3\pi i/4})$. The Bessel function $J_1(x)$ is defined in `BSJ1` (page 86). Function `BER1` is based on the work of Burgoyne (1963).

If $|x| > 119$, then NaN (not a number) is returned.

Example

In this example, $\text{ber}_1(0.4)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       BER1, VALUE, X
      EXTERNAL   BER1, UMACH
C                                 Compute
```

```
      X     = 0.4
      VALUE = BER1(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BER1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BER1( 0.400) = -0.144
```

BEI1/DBEI1 (Single/Double precision)

Evaluate the Kelvin function of the first kind, bei, of order one.

Usage

BEI1(X)

Arguments

X — Argument for which the function value is desired. (Input)

BEI1 — Function value. (Output)

Algorithm

The Kelvin function $\mathrm{bei}_1(x)$ is defined to be $\Im J_1(xe^{3\pi i/4})$. The Bessel function $J_1(x)$ is defined in BSJ1 (page 86). Function BEI1 is based on the work of Burgoyne (1963).

If $|x| > 119$, then NaN (not a number) is returned.

Example

In this example, $\mathrm{bei}_1(0.4)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       BEI1, VALUE, X
      EXTERNAL   BEI1, UMACH
C                                  Compute
      X     = 0.4
      VALUE = BEI1(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BEI1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BEI1( 0.400) = 0.139
```

AKER1/DKER1 (Single/Double precision)

Evaluate the Kelvin function of the second kind, ker, of order one.

Usage

`AKER1(X)`

Arguments

X **—** Argument for which the function value is desired. (Input)
It must be nonnegative.

AKER1 — Function value. (Output)

Algorithm

The modified Kelvin function $\mathrm{ker}_1(x)$ is defined to be $e^{-\pi i/2}\Re K_1(xe^{\pi i/4})$. The Bessel function $K_1(x)$ is defined in `BSK1` (page 93). Function `AKER1` is based on the work of Burgoyne (1963).

If $x < 0$, then NaN (not a number) is returned. If $x \geq 119$, then zero is returned.

Example

In this example, $\mathrm{ker}_1(0.4)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       AKER1, VALUE, X
      EXTERNAL   AKER1, UMACH
C                                  Compute
      X     = 0.4
      VALUE = AKER1(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKER1(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKER1( 0.400) = -1.882
```

AKEI1/DKEI1 (Single/Double precision)

Evaluate the Kelvin function of the second kind, kei, of order one.

Usage

`AKEI1(X)`

Arguments

X — Argument for which the function value is desired. (Input)
It must be nonnegative.

AKEI1 — Function value. (Output)

Algorithm

The modified Kelvin function $\mathrm{kei}_1(x)$ is defined to be $e^{-\pi i/2}\Im K_1(xe^{\pi i/4})$. The Bessel function $K_1(x)$ is defined in `BSK1` (page 93). Function `AKER1` is based on the work of Burgoyne (1963).

If $x < 0$, then NaN (not a number) is returned. If $x \geq 119$, then zero is returned.

Example

In this example, $\mathrm{kei}_1(0.4)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AKER1, VALUE, X
      EXTERNAL   AKER1, UMACH
C                                 Compute
      X     = 0.4
      VALUE = AKER1(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKER1(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKER1( 0.400) = -1.882
```

Chapter 8: Airy Functions

Routines

Evaluate Ai(x) .... AI 133
Evaluate Bi(x) .... BI 134
Evaluate Ai′(x) .... AID 135
Evaluate Bi′(x) .... BID 136
Evaluate exponentially scaled Ai(x) .... AIE 137
Evaluate exponentially scaled Bi(x) .... BIE 138
Evaluate exponentially scaled Ai′(x) .... AIDE 139
Evaluate exponentially scaled Bi′(x) .... BIDE 140

AI/DAI (Single/Double precision)

Evaluate the Airy function.

Usage

`AI(X)`

Arguments

X — Argument for which the Airy function is desired. (Input)

AI — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is greater than `XMAX`, where `XMAX` = $(-3/2 \ln(\texttt{AMACH}(1)))^{2/3}$.

Algorithm

The Airy function Ai(x) is defined to be

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{1}{3}t^3\right)dt = \sqrt{\frac{x}{3\pi^2}}K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$$

The Bessel function $K_\nu(x)$ is defined in `BSKS` (page 109).

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision. Finally, x should be less than $x_{\max}$ so the answer does not underflow. Very approximately, $x_{\max} = \{-1.5 \ln s\}^{2/3}$, where s = `AMACH(1)`, the smallest representable positive number. If underflows are a problem for large x, then the exponentially scaled routine `AIE` (page 137) should be used.

Example

In this example, Ai(–4.9) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AI, VALUE, X
      EXTERNAL   AI, UMACH
C                                 Compute
      X     = -4.9
      VALUE = AI(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AI(', F6.3, ') = ', F6.3)
      END
```

Output

```
AI(-4.900) = 0.375
```

BI/DBI (Single/Double precision)

Evaluate the Airy function of the second kind.

Usage

`BI(X)`

Arguments

X — Argument for which the Airy function value is desired. (Input)

BI — Function value. (Output)

Algorithm

The Airy function of the second kind Bi(x) is defined to be

$$\mathrm{Bi}(x)=\frac{1}{\pi}\int_0^\infty \exp\left(xt-\frac{1}{3}t^3\right)dt+\frac{1}{\pi}\int_0^\infty \sin\left(xt+\frac{1}{3}t^3\right)dt$$

It can also be expressed in terms of modified Bessel functions of the first kind, $I_\nu(x)$, and Bessel functions of the first kind, $J_\nu(x)$ (see `BSIS`, page 106, and `BSJS`, page 103):

$$\mathrm{Bi}(x)=\sqrt{\frac{x}{3}}\left[I_{-1/3}\left(\frac{2}{3}x^{3/2}\right)+I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right] \qquad \text{for } x>0$$

and

$$\mathrm{Bi}(x)=\sqrt{-\frac{x}{3}}\left[J_{-1/3}\left(\frac{2}{3}|x|^{3/2}\right)-J_{1/3}\left(\frac{2}{3}|x|^{3/2}\right)\right] \quad \text{for } x<0$$

Let ε = `AMACH`(4), the machine precision. If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, the answer will be less accurate than half precision. In addition, x should not be so large that $\exp[(2/3)x^{3/2}]$ overflows. If overflows are a problem, consider using the exponentially scaled form of the Airy function of the second kind, `BIE` (page 138), instead.

Example

In this example, Bi(–4.9) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       BI, VALUE, X
      EXTERNAL   BI, UMACH
C                                  Compute
      X     = -4.9
      VALUE = BI(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BI(', F6.3, ') = ', F6.3)
      END
```

Output

```
BI(-4.900) = -0.058
```

AID/DAID (Single/Double precision)

Evaluate the derivative of the Airy function.

Usage

`AID(X)`

Arguments

X — Argument for which the Airy function value is desired. (Input)

AID — Function value. (Output)

Comments

Informational error

Type	Code	
2	1	The function underflows because X is greater than XMAX, where XMAX = –3/2 ln(AMACH(1)).

Algorithm

The function $\mathrm{Ai}'(x)$ is defined to be the derivative of the Airy function, $\mathrm{Ai}(x)$ (see AI, page 133).

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, the answer will be less accurate than half precision. Here, ε = AMACH(4) is the machine precision. Finally, x should be less than $x_{\max}$ so that the answer does not underflow. Very approximately, $x_{\max} = \{-1.5 \ln s\}$, where s = AMACH(1), the smallest representable positive number. If underflows are a problem for large x, then the exponentially scaled routine AIDE (page 139) should be used.

Example

In this example, $\mathrm{Ai}'(-4.9)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       AID, VALUE, X
      EXTERNAL   AID, UMACH
C                                  Compute
      X     = -4.9
      VALUE = AID(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AID(', F6.3, ') = ', F6.3)
      END
```

Output

```
AID(-4.900) = 0.147
```

BID/DBID (Single/Double precision)

Evaluate the derivative of the Airy function of the second kind.

Usage

BID(X)

Arguments

X — Argument for which the Airy function value is desired. (Input)

BID — Function value. (Output)

Algorithm

The function Bi′(x) is defined to be the derivative of the Airy function of the second kind, Bi(x) (see `BI`, page 134).

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, the answer will be less accurate than half precision. In addition, x should not be so large that $\exp[(2/3)x^{3/2}]$ overflows. If overflows are a problem, consider using `BIDE` (page 140) instead. Here, ε = `AMACH`(4) is the machine precision.

Example

In this example, Bi′(–4.9) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       BID, VALUE, X
      EXTERNAL   BID, UMACH
C                                  Compute
      X     = -4.9
      VALUE = BID(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BID(', F6.3, ') = ', F6.3)
      END
```

Output

```
BID(-4.900) = 0.827
```

AIE/DAIE (Single/Double precision)

Evaluate the exponentially scaled Airy function.

Usage

`AIE(X)`

Arguments

X — Argument for which the Airy function value is desired. (Input)

AIE — Function value. (Output)
The Airy function for negative arguments and the exponentially scaled Airy function, e^{ζ}Ai(`X`), for positive arguments where

$$\zeta = \frac{2}{3} X^{3/2}$$

Algorithm

The exponentially scaled Airy function is defined to be

$$\mathrm{AIE}(x) = \begin{cases} \mathrm{Ai}(x) & \text{if } x \le 0 \\ e^{[2/3]x^{3/2}} \mathrm{Ai}(x) & \text{if } x > 0 \end{cases}$$

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, then the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision.

Example

In this example, `AIE`(0.49) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AIE, VALUE, X
      EXTERNAL   AIE, UMACH
C                                 Compute
      X     = 0.49
      VALUE = AIE(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AIE(', F6.3, ') = ', F6.3)
      END
```

Output

```
AIE( 0.490) = 0.294
```

BIE/DBIE (Single/Double precision)

Evaluate the exponentially scaled Airy function of the second kind.

Usage

`BIE(X)`

Arguments

X — Argument for which the Airy function value is desired. (Input)

BIE — Function value. (Output)
The Airy function of the second kind for negative arguments and the exponentially scaled Airy function of the second kind, e^{ζ}Bi(`X`), for positive arguments where

$$\zeta = -\tfrac{2}{3} X^{3/2}$$

Algorithm

The exponentially scaled Airy function of the second kind is defined to be

$$\mathrm{BIE}(x) = \begin{cases} \mathrm{Bi}(x) & \text{if } x \le 0 \\ e^{-[2/3]x^{3/2}} \mathrm{Bi}(x) & \text{if } x > 0 \end{cases}$$

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, then the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision.

Example

In this example, `BIE`(0.49) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       BIE, VALUE, X
      EXTERNAL   BIE, UMACH
C                                  Compute
      X     = 0.49
      VALUE = BIE(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BIE(', F6.3, ') = ', F6.3)
      END
```

Output

```
BIE( 0.490) = 0.675
```

AIDE/DAIDE (Single/Double precision)

Evaluate the exponentially scaled derivative of the Airy function.

Usage

`AIDE(X)`

Arguments

X — Argument for which the Airy function value is desired. (Input)

AIDE — Function value. (Output)
The derivative of the Airy function for negative arguments and the exponentially scaled derivative of the Airy function, $e^{\zeta}\mathrm{Ai}'(X)$, for positive arguments where

$$\zeta = -\frac{2}{3} X^{3/2}$$

Algorithm

The exponentially scaled derivative of the Airy function is defined to be

$$\mathrm{AIDE}(x) = \begin{cases} \mathrm{Ai}'(x) & \text{if } x \le 0 \\ e^{[2/3]x^{3/2}}\,\mathrm{Ai}'(x) & \text{if } x > 0 \end{cases}$$

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, then the answer will be less accurate than half precision. Here, ε = AMACH(4) is the machine precision.

Example

In this example, AIDE(0.49) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       AIDE, VALUE, X
      EXTERNAL   AIDE, UMACH
C                                  Compute
      X     = 0.49
      VALUE = AIDE(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AIDE(', F6.3, ') = ', F6.3)
      END
```

Output

```
AIDE( 0.490) = -0.284
```

BIDE/DBIDE (Single/Double precision)

Evaluate the exponentially scaled derivative of the Airy function of the second kind.

Usage

BIDE(X)

Arguments

X — Argument for which the Airy function value is desired. (Input)

BIDE — Function value. (Output)
The derivative of the Airy function of the second kind for negative arguments and the exponentially scaled derivative of the Airy function of the second kind, $e^{\zeta}\mathrm{Bi}'$ (X), for positive arguments where

$$\zeta = -\frac{2}{3}X^{3/2}$$

Algorithm

The exponentially scaled derivative of the Airy function of the second kind is defined to be

$$\mathrm{BIDE}(x) = \begin{cases} \mathrm{Bi}'(x) & \text{if } x \le 0 \\ e^{-[2/3]x^{3/2}}\,\mathrm{Bi}'(x) & \text{if } x > 0 \end{cases}$$

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, then the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision.

Example

In this example, `BIDE`(0.49) is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       BIDE, VALUE, X
      EXTERNAL   BIDE, UMACH
C                                  Compute
      X     = 0.49
      VALUE = BIDE(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BIDE(', F6.3, ') = ', F6.3)
      END
```

Output

```
BIDE( 0.490) = 0.430
```

Chapter 9: Elliptic Integrals

Routines

Evaluate the complete elliptic integral of the first kind, $K(x)$...... ELK 145
Evaluate the complete elliptic integral of the second kind, $E(x)$........ ELE 147
Evaluate Carlson's elliptic integral of the first kind, $R_F(x, y, z)$........ ELRF 148
Evaluate Carlson's elliptic integral of the second kind, $R_D(x, y, z)$........ ELRD 149
Evaluate Carlson's elliptic integral of the third kind, $R_J(x, y, z)$........ ELRJ 150
Evaluate a special case of Carlson's elliptic integral, $R_C(x, y, z)$........ ELRC 151

Usage Notes

The notation used in this chapter follows that of Abramowitz and Stegun (1964) and Carlson (1979).

The complete elliptic integral of the first kind is

$$K(m) = \int_0^{\pi/2} \left(1 - m \sin^2 \theta\right)^{-1/2} d\theta$$

and the complete elliptic integral of the second kind is

$$E(m) = \int_0^{\pi/2} \left(1 - m \sin^2 \theta\right)^{1/2} d\theta$$

Instead of the *parameter* m, the *modular* angle α is sometimes used with $m = \sin^2 \alpha$. Also used is the *modulus* k with $k^2 = m$.

$$E(k) = \int_0^{\pi/2} \left(1 - k^2 \sin^2 \theta\right)^{1/2} d\theta$$
$$= R_F\left(0, 1 - k^2, 1\right) - \frac{1}{3} k^2 R_D\left(0, 1 - k^2, 1\right)$$

Carlson Elliptic Integrals

The Carlson elliptic integrals are defined by Carlson (1979) as follows:

$$R_F(x, y, z) = \frac{1}{2} \int_0^\infty \frac{dt}{\left[(t+x)(t+y)(t+z)\right]^{1/2}}$$

$$R_C(x, y) = \frac{1}{2} \int_0^\infty \frac{dt}{\left[(t+x)(t+y)^2\right]^{1/2}}$$

$$R_J(x, y, z, \rho) = \frac{3}{2} \int_0^\infty \frac{dt}{\left[(t+x)(t+y)(t+z)(t+\rho)^2\right]^{1/2}}$$

$$R_D(x, y, z) = \frac{3}{2} \int_0^\infty \frac{dt}{\left[(t+x)(t+y)(t+z)^3\right]^{1/2}}$$

The standard Legendre elliptic integrals can be written in terms of the Carlson functions as follows (these relations are from Carlson (1979)):

$$F(\phi, k) = \int_0^\phi \left(1 - k^2 \sin^2 \theta\right)^{-1/2} d\theta$$
$$= (\sin \phi) R_F\left(\cos^2 \phi, 1 - k^2 \sin^2 \phi, 1\right)$$

$$E(\phi, k) = \int_0^\phi \left(1 - k^2 \sin^2 \theta\right)^{1/2} d\theta$$
$$= (\sin \phi) R_F\left(\cos^2 \phi, 1 - k^2 \sin^2 \phi, 1\right) - \frac{1}{3} k^2 (\sin \phi)^3 R_D\left(\cos^2 \phi, 1 - k^2 \sin^2 \phi, 1\right)$$

$$\Pi(\phi, k, n) = \int_0^\phi \left(1 + n \sin^2 \theta\right)^{-1} \left(1 - k^2 \sin^2 \theta\right)^{-1/2} d\theta$$
$$= (\sin \phi) R_F\left(\cos^2 \phi, 1 - k^2 \sin^2 \phi, 1\right) - \frac{n}{3} k^2 (\sin \phi)^3 R_D\left(\cos^2 \phi, 1 - k^2 \sin^2 \phi, 1 + n \sin^2 \phi\right)$$

$$D(\phi, k, n) = \int_0^\phi \sin^2 \theta \left(1 - k^2 \sin^2 \theta\right)^{-1/2} d\theta$$
$$= \frac{1}{3} (\sin \phi)^3 R_D\left(\cos^2 \phi, 1 - k^2 \sin^2 \phi, 1\right)$$

$$K(k) = \int_0^{\pi/2} \left(1 - k^2 \sin^2 \theta\right)^{-1/2} d\theta$$
$$= R_F\left(0, 1 - k^2, 1\right)$$
$$E(k) = \int_0^{\pi/2} \left(1 - k^2 \sin^2 \theta\right)^{1/2} d\theta$$
$$= R_F\left(0, 1 - k^2, 1\right) - \frac{1}{3} k^2 R_D\left(0, 1 - k^2, 1\right)$$

The function $R_C(x, y)$ is related to inverse trigonometric and inverse hyperbolic functions.

$$\ln x = (x-1)\ R_c\left[\left(\frac{1+x}{2}\right), x\right] \qquad 0 < x < \infty$$
$$\sin^{-1} x = x R_c\left(1 - x^2, 1\right) \qquad -1 \le x \le 1$$
$$\sinh^{-1} x = x R_c\left(1 + x^2, 1\right) \qquad -\infty < x < \infty$$
$$\cos^{-1} x = \sqrt{1 - x^2}\ R_c\left(x^2, 1\right) \qquad 0 \le x \le 1$$
$$\cosh^{-1} x = \sqrt{x^2 - 1}\ R_c\left(x^2, 1\right) \qquad 1 \le x < \infty$$
$$\tan^{-1} x = x R_c\left(1, 1 + x^2\right) \qquad -\infty < x < \infty$$
$$\tanh^{-1} x = x R_c\left(1, 1 - x^2\right) \qquad -1 < x < 1$$
$$\cot^{-1} x = R_c\left(x^2, x^2 + 1\right) \qquad 0 < x < \infty$$
$$\coth^{-1} x = R_c\left(x^2, x^2 - 1\right) \qquad 1 < x < \infty$$

ELK/DELK (Single/Double precision)

Evaluate the complete elliptic integral of the kind K(x).

Usage

`ELK(X)`

Arguments

X — Argument for which the function value is desired. (Input)
`X` must be greater than or equal to 0 and less than 1.

ELK — Function value. (Output)

Algorithm

The complete elliptic integral of the first kind is defined to be

$$K(x) = \int_0^{\pi/2} \frac{d\,\theta}{\left[1 - x\sin^2\theta\right]^{1/2}} \quad \text{for } 0 \le x < 1$$

The argument x must satisfy $0 \le x < 1$; otherwise, `ELK` is set to b = `AMACH`(2), the largest representable floating-point number.

The function $K(x)$ is computed using the routine `ELRF` (page 148) and the relation $K(x) = R_F(0, 1 - x, 1)$.

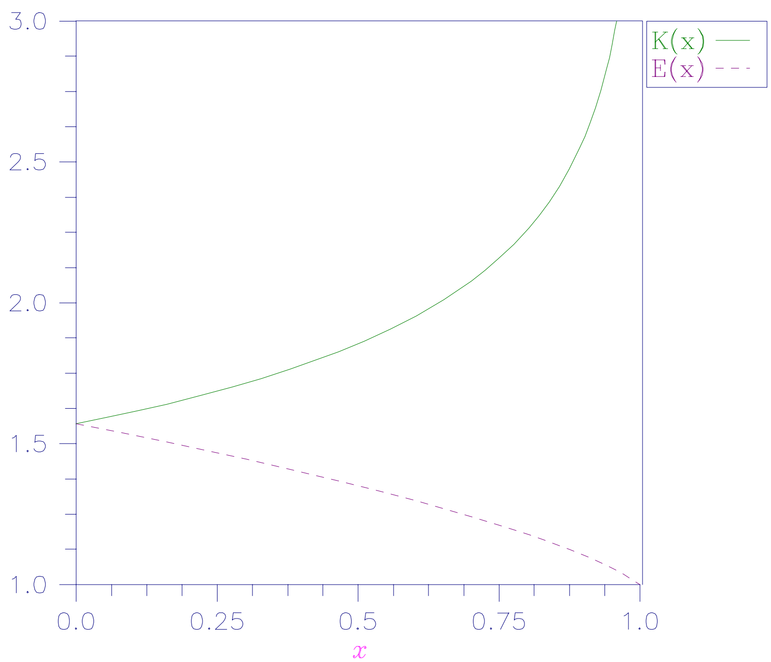

Figure 9-1 Plot of $K(x)$ and $E(x)$

Example

In this example, $K(0)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ELK, VALUE, X
      EXTERNAL   ELK, UMACH
C                                 Compute
      X     = 0.0
      VALUE = ELK(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
```

```
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ELK(', F6.3, ') = ', F6.3)
      END
```

Output

```
ELK( 0.000) = 1.571
```

ELE/DELE (Single/Double precision)

Evaluate the complete elliptic integral of the second kind $E(x)$.

Usage

`ELE(X)`

Arguments

X — Argument for which the function value is desired. (Input)
`X` must be greater than or equal to 0 and less than or equal to 1.

ELE — Function value. (Output)

Algorithm

The complete elliptic integral of the second kind is defined to be

$$E(x) = \int_0^{\pi/2} \left[1 - x\sin^2\theta\right]^{1/2} d\theta \quad \text{for } 0 \le x < 1$$

The argument x must satisfy $0 \le x < 1$; otherwise, `ELE` is set to b = `AMACH`(2), the largest representable floating-point number.

The function $E(x)$ is computed using the routines `ELRF`, page 148, and `ELRD`, page 149. The computation is done using the relation

$$E(x) = R_F(0, 1-x, 1) - \frac{x}{3} R_D(0, 1-x, 1)$$

For a plot of $E(x)$, see Figure 9.1 on page 146.

Example

In this example, $E(0.33)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       ELE, VALUE, X
      EXTERNAL   ELE, UMACH
C                                  Compute
      X     = 0.33
      VALUE = ELE(X)
C                                  Print the results
      CALL UMACH (2, NOUT)
```

```
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ELE(', F6.3, ') = ', F6.3)
      END
```

Output

```
ELE( 0.330) = 1.432
```

ELRF/DELRF (Single/Double precision)

Evaluate Carlson's incomplete elliptic integral of the first kind R_F(X, Y, Z).

Usage

ELRF(X, Y, Z)

Arguments

X — First variable of the incomplete elliptic integral. (Input)
It must be nonnegative

Y — Second variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

Z — Third variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

ELRF — Function value. (Output)

Algorithm

The Carlson's complete elliptic integral of the first kind is defined to be

$$R_F(x,\ y,\ z) = \frac{1}{2}\int_0^\infty \frac{dt}{\left[(t+x)(t+y)(t+z)\right]^{1/2}}$$

The arguments must be nonnegative and less than or equal to $b/5$. In addition, $x + y$, $x + z$, and $y + z$ must be greater than or equal to $5s$. Should any of these conditions fail, ELRF is set to b. Here, b = AMACH(2) is the largest and s = AMACH(1) is the smallest representable floating-point number.

The function ELRF is based on the code by Carlson and Notis (1981) and the work of Carlson (1979).

Example

In this example, $R_F(0, 1, 2)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ELRF, VALUE, X, Y, Z
      EXTERNAL   ELRF, UMACH
C                                   Compute
```

```
      X     = 0.0
      Y     = 1.0
      Z     = 2.0
      VALUE = ELRF(X, Y, Z)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, Y, Z, VALUE
99999 FORMAT (' ELRF(', F6.3, ',', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
ELRF( 0.000, 1.000, 2.000) = 1.311
```

ELRD/DELRD (Single/Double precision)

Evaluate Carlson's incomplete elliptic integral of the second kind R_D(X, Y, Z).

Usage

ELRD(X, Y, Z)

Arguments

X — First variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

Y — Second variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

Z — Third variable of the incomplete elliptic integral. (Input)
It must be positive.

ELRD — Function value. (Output)

Algorithm

The Carlson's complete elliptic integral of the second kind is defined to be

$$R_D(x, y, z) = \frac{3}{2}\int_0^{\infty} \frac{dt}{\left[(t+x)(t+y)(t+z)^3\right]^{1/2}}$$

The arguments must be nonnegative and less than or equal to $0.69(-\ln \varepsilon)^{1/9}\, s^{-2/3}$ where ε = AMACH(4) is the machine precision, s = AMACH(1) is the smallest representable positive number. Furthermore, $x + y$ and z must be greater than $\max\{3s^{2/3}, 3/b^{2/3}\}$, where b = AMACH(2) is the largest floating-point number. If any of these conditions are false, then ELRD is set to b.

The function ELRD is based on the code by Carlson and Notis (1981) and the work of Carlson (1979).

Example

In this example, $R_D(0, 2, 1)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       ELRD, VALUE, X, Y, Z
      EXTERNAL   ELRD, UMACH
C                                  Compute
      X     = 0.0
      Y     = 2.0
      Z     = 1.0
      VALUE = ELRD(X, Y, Z)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, Y, Z, VALUE
99999 FORMAT (' ELRD(', F6.3, ',', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
ELRD( 0.000, 2.000, 1.000) = 1.797
```

ELRJ/DELRJ (Single/Double precision)

Evaluate Carlson's incomplete elliptic integral of the third kind R_J(X, Y, Z, RHO)

Usage

`ELRJ(X, Y, Z, RHO)`

Arguments

X — First variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

Y — Second variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

Z — Third variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

RHO — Fourth variable of the incomplete elliptic integral. (Input)
It must be positive.

ELRJ — Function value. (Output)

Algorithm

The Carlson's complete elliptic integral of the third kind is defined to be

$$R_J(x, y, z, \rho) = \frac{3}{2}\int_0^\infty \frac{dt}{\left[(t+x)(t+y)(t+z)(t+\rho)^2\right]^{1/2}}$$

The arguments must be nonnegative. In addition, $x + y$, $x + z$, $y + z$ and ρ must be greater than or equal to $(5s)^{1/3}$ and less than or equal to $.3(b/5)^{1/3}$, where s = AMACH(1) is the smallest representable floating-point number. Should any of these conditions fail, ELRF is set to b = AMACH(2), the largest floating-point number.

The function ELRJ is based on the code by Carlson and Notis (1981) and the work of Carlson (1979).

Example

In this example, $R_J(2, 3, 4, 5)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ELRJ, RHO, VALUE, X, Y, Z
      EXTERNAL   ELRJ, UMACH
C                                 Compute
      X     = 2.0
      Y     = 3.0
      Z     = 4.0
      RHO   = 5.0
      VALUE = ELRJ(X, Y, Z, RHO)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, Y, Z, RHO, VALUE
99999 FORMAT (' ELRJ(', F6.3, ',', F6.3, ',', F6.3, ',', F6.3,
     &     ') = ', F6.3)
      END
```

Output

```
ELRJ( 2.000, 3.000, 4.000, 5.000) = 0.143
```

ELRC/DELRC (Single/Double precision)

Evaluate an elementary integral from which inverse circular functions, logarithms and inverse hyperbolic functions can be computed.

Usage

ELRC(X, Y)

Arguments

X — First variable of the incomplete elliptic integral. (Input)
It must be nonnegative and satisfy the conditions given in Comments.

Y — Second variable of the incomplete elliptic integral. (Input)
It must be positive and satisfy the conditions given in Comments.

ELRC — Function value. (Output)

Comments

The sum X + Y must be greater than or equal to ARGMIN and both X and Y must be less than or equal to ARGMAX. ARGMIN = s * 5 and ARGMAX = b/5, where s is the machine minimum (AMACH(1)) and b is the machine maximum (AMACH(2)).

Algorithm

The special case of Carlson's complete elliptic integral of the first kind is defined to be

$$R_C(x, y) = \frac{1}{2}\int_0^\infty \frac{dt}{\left[(t+x)(t+y)^2\right]^{1/2}}$$

The argument x must be nonnegative, y must be positive, and $x + y$ must be less than or equal to $b/5$ and greater than or equal to $5s$. If any of these conditions are false, then ELRC is set to b. Here, b = AMACH(2) is the largest and s = AMACH(1) is the smallest representable floating-point number.

The function ELRF is based on the code by Carlson and Notis (1981) and the work of Carlson (1979).

Example

In this example, $R_C(2.25, 2.0)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       ELRF, VALUE, X, Y, Z
      EXTERNAL   ELRF, UMACH
C                                 Compute
      X     = 0.0
      Y     = 1.0
      Z     = 2.0
      VALUE = ELRF(X, Y, Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, Y, Z, VALUE
99999 FORMAT (' ELRF(', F6.3, ',', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
ELRF( 0.000, 1.000, 2.000) = 1.311
```

Chapter 10: Elliptic and Related Functions

Routines

10.1. Weierstrass Elliptic and Related Functions

Lemninscatic case .... CWPL 154
Lemninscatic case derivative ....CWPLD 155
Equianharmonic case .... CWPQ 156
Equianharmonic case derivative ....CWPQD 157

10.2. Jacobi Elliptic Functions

Jacobi function sn(x, m) (real argument) ....EJSN 158
Jacobi function sn(z, m) (complex argument)......................CEJSN 159
Jacobi function cn(x, m) (real argument) .... EJCN 160
Jacobi function cn(z, m) (complex argument)......................CEJCN 162
Jacobi function dn(x, m) (real argument) .... EJDN 163
Jacobi function dn(z, m) (complex argument)......................CEJDN 164

Usage Notes

Elliptic functions are doubly periodic, single-valued complex functions of a single variable that are analytic, except at a finite number of poles. Because of the periodicity, we need consider only the fundamental period parallelogram. The irreducible number of poles, counting multiplicities, is the *order* of the elliptic function. The simplest, non-trivial, elliptic functions are of order two.

The Weierstrass elliptic functions, $\wp(z, \omega, \omega')$ have a double pole at $z = 0$ and so are of order two. Here, 2ω and $2\omega'$ are the periods.

The Jacobi elliptic functions each have two simple poles and so are also of order two. The period of the functions is as follows:

Function	**Periods**
sn(x, m)	$4K(m)\quad 2iK'(m)$

cn(x, m)	$4K(m)$	$4iK'(m)$
dn(x, m)	$2K(m)$	$4iK'(m)$

The function $K(m)$ is the complete elliptic integral, see `ELK` (page 145), and $K'(m) = K(1 - m)$.

CWPL/ZWPL (Single/Double precision)

Evaluate the Weierstrass' $\wp$ function in the lemniscatic case for complex argument with unit period parallelogram.

Usage

`CWPL(Z)`

Arguments

Z — Complex argument for which the function value is desired. (Input)

CWPL — Complex function value. (Output)

Algorithm

The Weierstrass' $\wp$ function, $\wp(z) = \wp(z \mid \omega, \omega')$, is an elliptic function of order two with periods 2ω and $2\omega'$ and a double pole at $z = 0$. `CWPL(Z)` computes $\wp(z \mid \omega, \omega')$ with $2\omega = 1$ and $2\omega' = i$.

The input argument is first reduced to the fundamental parallelogram of all z satisfying $-1/2 \le \Re z \le 1/2$ and $-1/2 \le \Im z \le 1/2$. Then, a rational approximation is used.

All arguments are valid with the exception of the lattice points $z = m + ni$, which are the poles of `CWPL`. If the argument is a lattice point, then b = `AMACH`(2) , the largest floating-point number, is returned. If the argument has modulus greater than $10\varepsilon^{-1}$, then NaN (not a number) is returned. Here, ε = `AMACH`(4) is the machine precision.

Function `CWPL` is based on code by Eckhardt (1980). Also, see Eckhardt (1977).

Example

In this example, $\wp(0.25 + 0.25i)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      COMPLEX    CWPL, VALUE, Z
      EXTERNAL   CWPL, UMACH
C                                  Compute
      Z     = (0.25, 0.25)
      VALUE = CWPL(Z)
C                                  Print the results
      CALL UMACH (2, NOUT)
```

```
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CWPL(', F6.3, ',', F6.3, ') = (',
     &      F6.3, ',', F6.3, ')')
      END
```

Output

```
CWPL( 0.250, 0.250) = ( 0.000,-6.875)
```

CWPLD/ZWPLD (Single/Double precision)

Evaluate the first derivative of the Weierstrass' $\wp$ function in the lemniscatic case for complex argument with unit period parallelogram.

Usage

CWPLD(Z)

Arguments

Z — Complex argument for which the function value is desired. (Input)

CWPLD — Complex function value. (Output)

Algorithm

The Weierstrass' $\wp$ function, $\wp(z) = \wp(z \mid \omega, \omega')$, is an elliptic function of order two with periods 2ω and $2\omega'$ and a double pole at $z = 0$. CWPLD(Z) computes the derivative of $\wp(z \mid \omega, \omega')$ with $2\omega = 1$ and $2\omega' = i$. CWPL, page 154, computes $\wp(z \mid \omega, \omega')$.

The input argument is first reduced to the fundamental parallelogram of all z satisfying $-1/2 \le \Re z \le 1/2$ and $-1/2 \le \Im z \le 1/2$. Then, a rational approximation is used.

All arguments are valid with the exception of the lattice points $z = m + ni$, which are the poles of CWPL. If the argument is a lattice point, then b = AMACH(2), the largest floating-point number, is returned.

Function CWPLD is based on code by Eckhardt (1980). Also, see Eckhardt (1977).

Example

In this example, $\wp(0.25 + 0.25i)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      COMPLEX    CWPLD, VALUE, Z
      EXTERNAL   CWPLD, UMACH
C                                  Compute
      Z     = (0.25, 0.25)
      VALUE = CWPLD(Z)
```

```
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CWPLD(', F6.3, ',', F6.3, ') = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CWPLD( 0.250, 0.250) = (36.054,36.054)
```

CWPQ/ZWPQ (Single/Double precision)

Evaluate the Weierstrass' $\wp$ function in the equianharmonic case for complex argument with unit period parallelogram.

Usage

`CWPQ(Z)`

Arguments

Z — Complex argument for which the function value is desired. (Input)

CWPQ — Complex function value. (Output)

Algorithm

The Weierstrass' $\wp$ function, $\wp(z) = \wp(z \mid \omega, \omega')$, is an elliptic function of order two with periods 2ω and $2\omega'$ and a double pole at $z = 0$. `CWPQ(Z)` computes $\wp(z \mid \omega, \omega')$ with

$$4\omega = 1 - i\sqrt{3} \text{ and } 4\omega' = 1 + i\sqrt{3}$$

The input argument is first reduced to the fundamental parallelogram of all z satisfying

$$-1/2 \le \Re z \le 1/2 \text{ and } -\sqrt{3}/4 \le \Im z \le \sqrt{3}/4$$

Then, a rational approximation is used.

All arguments are valid with the exception of the lattice points

$$z = m\left(1 - i\sqrt{3}\right) + n\left(1 + i\sqrt{3}\right)$$

which are the poles of `CWPQ`. If the argument is a lattice point, then b = `AMACH`(2), the largest floating-point number, is returned. If the argument has modulus greater than $10\varepsilon^{-1}$, then NaN (not a number) is returned. Here,
ε = `AMACH`(4) is the machine precision.

Function `CWPQ` is based on code by Eckhardt (1980). Also, see Eckhardt (1977).

Example

In this example, $\wp(0.25 + 0.14437567i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CWPQ, VALUE, Z
      EXTERNAL   CWPQ, UMACH
C                                 Compute
      Z     = (0.25, 0.14437567)
      VALUE = CWPQ(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CWPQ(', F6.3, ',', F6.3, ') = (',
     &     F7.3, ',', F7.3, ')')
      END
```

Output

```
CWPQ( 0.250, 0.144) = ( 5.895,-10.216)
```

CWPQD/ZWPQD (Single/Double precision)

Evaluate the first derivative of the Weierstrass' $\wp$ function in the equianharmonic case for complex argument with unit period parallelogram.

Usage

`CWPQD(Z)`

Arguments

Z — Complex argument for which the function value is desired. (Input)

CWPQD — Complex function value. (Output)

Algorithm

The Weierstrass' $\wp$ function, $\wp(z) = \wp(z \mid \omega, \omega')$, is an elliptic function of order two with periods 2ω and $2\omega'$ and a double pole at $z = 0$. `CWPQD(Z)` computes the derivative of $\wp(z \mid \omega, \omega')$ with

$$4\omega = 1 - i\sqrt{3} \text{ and } 4\omega' = 1 + i\sqrt{3}$$

`CWPQ`, page 156, computes $\wp(z \mid \omega, \omega')$.

The input argument is first reduced to the fundamental parallelogram of all z satisfying

$$-1/2 \le \Re z \le 1/2 \text{ and } -\sqrt{3}/4 \le \Im z \le \sqrt{3}/4$$

Then, a rational approximation is used.

All arguments are valid with the exception of the lattice points

$$z = m\left(1 - i\sqrt{3}\right) + n\left(1 + i\sqrt{3}\right)$$

which are the poles of `CWPQ`. If the argument is a lattice point, then b = `AMACH`(2), the largest floating-point number, is returned.

Function `CWPQD` is based on code by Eckhardt (1980). Also, see Eckhardt (1977).

Example

In this example, $\wp(0.25 + 0.14437567i)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      COMPLEX    CWPQD, VALUE, Z
      EXTERNAL   CWPQD, UMACH
C                                 Compute
      Z     = (0.25, 0.14437567)
      VALUE = CWPQD(Z)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CWPQD(', F6.3, ',', F6.3, ') = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CWPQD( 0.250, 0.144) = ( 0.028,85.934)
```

EJSN/DEJSN (Single/Double precision)

Evaluate the Jacobi elliptic function sn(x, m).

Usage

`EJSN(X, AM)`

Arguments

X — Argument for which the function value is desired. (Input)

AM — Parameter of the elliptic function ($m = k^2$). (Input)

EJSN — Function value. (Output)

Comments

Informational errors

Type	Code	
3	2	The result is accurate to less than one half precision because |X| is too large.

3	5	Landen transform did not converge. Result may not be accurate. This should never occur.

Algorithm

The Jacobi elliptic function sn(x, m) = sin ϕ, where the amplitude ϕ is defined by the following:

$$x = \int_0^{\phi} \frac{d\theta}{\left(1 - m \sin^2 \theta\right)^{1/2}}$$

The function sn(x, m) is computed by first applying, if necessary, a Jacobi transformation so that the parameter, m, is between zero and one. Then, a descending Landen (Gauss) transform is applied until the parameter is small. The small parameter approximation is then applied.

Example

In this example, sn(1.5, 0.5) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AM, EJSN, VALUE, X
      EXTERNAL   EJSN, UMACH
C                                 Compute
      AM    = 0.5
      X     = 1.5
      VALUE = EJSN(X, AM)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, AM, VALUE
99999 FORMAT (' EJSN(', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
EJSN( 1.500, 0.500) = 0.968
```

CEJSN/ZEJSN (Single/Double precision)

Evaluate the complex Jacobi elliptic function sn(z, m).

Usage

CEJSN(Z, AM)

Arguments

Z — Complex argument for which the function value is desired. (Input)

AM — Real parameter of the elliptic function ($m = k^2$). (Input)

CEJSN — Complex function value. (Output)

Comments

Informational errors

Type	Code	
3	2	The result is accurate to less than one half precision because |REAL (Z)| is too large.
3	3	The result is accurate to less than one half precision because |AIMAG (Z)| is too large.
3	5	Landen transform did not converge. Result may not be accurate. This should never occur.

Algorithm

The Jacobi elliptic function sn(z, m) = sin ϕ, where the amplitude ϕ is defined by the following:

$$z = \int_0^{\phi} \frac{d\theta}{\left(1 - m\sin^2\theta\right)^{1/2}}$$

The function sn(z, m) is computed by first applying, if necessary, a Jacobi transformation so that the parameter, m, is between zero and one. Then, a descending Landen (Gauss) transform is applied until the parameter is small. The small parameter approximation is then applied.

Example

In this example, sn(1.5 + 0.3i, 0.5) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AM
      COMPLEX    CEJSN, VALUE, Z
      EXTERNAL   CEJSN, UMACH
C                                 Compute
      Z     = (1.5, 0.3)
      AM    = 0.5
      VALUE = CEJSN(Z, AM)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, AM, VALUE
99999 FORMAT (' CEJSN((', F6.3, ',', F6.3, '), ', F6.3, ') = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CEJSN(( 1.500, 0.300), 0.500) = ( 0.993, 0.054)
```

EJCN/DEJCN (Single/Double precision)

Evaluate the Jacobi elliptic function cn(x, m).

Usage

`EJCN(X, AM)`

Arguments

X — Argument for which the function value is desired. (Input)

AM — Parameter of the elliptic function ($m = k^2$). (Input)

EJCN — Function value. (Output)

Comments

Informational errors

Type	Code	
3	2	The result is accurate to less than one half precision because |X| is too large.
3	5	Landen transform did not converge. Result may not be accurate. This should never occur.

Algorithm

The Jacobi elliptic function cn(x, m) = cos ϕ, where the amplitude ϕ is defined by the following:

$$x = \int_0^{\phi} \frac{d\theta}{\left(1 - m\sin^2\theta\right)^{1/2}}$$

The function cn(x, m) is computed by first applying, if necessary, a Jacobi transformation so that the parameter, m, is between zero and one. Then, a descending Landen (Gauss) transform is applied until the parameter is small. The small parameter approximation is then applied.

Example

In this example, cn(1.5, 0.5) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AM, EJCN, VALUE, X
      EXTERNAL   EJCN, UMACH
C                                 Compute
      AM    = 0.5
      X     = 1.5
      VALUE = EJCN(X, AM)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, AM, VALUE
99999 FORMAT (' EJCN(', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
EJCN( 1.500, 0.500) = 0.250
```

CEJCN/ZEJCN (Single/Double precision)

Evaluate the complex Jacobi elliptic integral cn(z, m).

Usage

`CEJCN(Z, AM)`

Arguments

Z — Complex argument for which the function value is desired. (Input)

AM — Parameter of the elliptic integral ($m = k^2$). (Input)

CEJCN — Complex function value. (Output)

Comments

Informational errors

Type	Code	
3	2	The result is accurate to less than one half precision because |`REAL (Z)`| is too large.
3	3	The result is accurate to less than one half precision because |`AIMAG (Z)`| is too large.
3	5	Landen transform did not converge. Result may not be accurate. This should never occur.

Algorithm

The Jacobi elliptic function cn(z, m) = cos ϕ, where the amplitude ϕ is defined by the following:

$$z = \int_0^{\phi} \frac{d\theta}{\left(1 - m \sin^2 \theta\right)^{1/2}}$$

The function cn(z, m) is computed by first applying, if necessary, a Jacobi transformation so that the parameter, m, is between zero and one. Then, a descending Landen (Gauss) transform is applied until the parameter is small. The small parameter approximation is then applied.

Example

In this example, cn(1.5 + 0.3i, 0.5) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AM
      COMPLEX    CEJCN, VALUE, Z
```

```
      EXTERNAL   CEJCN, UMACH
C                                  Compute
      Z     = (1.5, 0.3)
      AM    = 0.5
      VALUE = CEJCN(Z, AM)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, AM, VALUE
99999 FORMAT (' CEJCN((', F6.3, ',', F6.3, '), ', F6.3, ') = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CEJCN(( 1.500, 0.300), 0.500) = ( 0.251,-0.212)
```

EJDN/DEJDN (Single/Double precision)

Evaluate the Jacobi elliptic function dn(x, m).

Usage

`EJDN(X, AM)`

Arguments

X — Argument for which the function value is desired. (Input)

AM — Parameter of the elliptic function ($m = k^2$). (Input)

EJDN — Function value. (Output)

Comments

Informational errors

Type	Code	
3	2	The result is accurate to less than one half precision because |X| is too large.
3	5	Landen transform did not converge. Result may not be accurate. This should never occur.

Algorithm

The Jacobi elliptic function $\mathrm{dn}(x, m) = (1 - m \sin^2 \phi)^{1/2}$, where the amplitude ϕ is defined by the following:

$$x = \int_0^{\phi} \frac{d\theta}{\left(1 - m \sin^2 \theta\right)^{1/2}}$$

The function dn(x, m) is computed by first applying, if necessary, a Jacobi transformation so that the parameter, m, is between zero and one. Then, a descending Landen (Gauss) transform is applied until the parameter is small. The small parameter approximation is then applied.

Example

In this example, dn(1.5, 0.5) is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       AM, EJDN, VALUE, X
      EXTERNAL   EJDN, UMACH
C                                 Compute
      AM    = 0.5
      X     = 1.5
      VALUE = EJDN(X, AM)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, AM, VALUE
99999 FORMAT (' EJDN(', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
EJDN( 1.500, 0.500) = 0.729
```

CEJDN/ZEJDN (Single/Double precision)

Evaluate the complex Jacobi elliptic integral dn(z, m).

Usage

`CEJDN(Z, AM)`

Arguments

Z — Complex argument for which the function value is desired. (Input)

AM — Parameter of the elliptic integral ($m = k^2$). (Input)

CEJDN — Complex function value. (Output)

Comments

Informational errors

Type	Code	
3	2	The result is accurate to less than one half precision because |`REAL (Z)`| is too large.
3	3	The result is accurate to less than one half precision because |`AIMAG (Z)`| is too large.
3	5	Landen transform did not converge. Result may not be accurate. This should never occur.

Algorithm

The Jacobi elliptic function $\mathrm{dn}(z, m) = (1 - m \sin^2 \phi)^{1/2})$, where the amplitude ϕ is defined by the following:

$$z = \int_0^{\phi} \frac{d\theta}{\left(1 - m \sin^2 \theta\right)^{1/2}}$$

The function $\mathrm{dn}(z, m)$ is computed by first applying, if necessary, a Jacobi transformation so that the parameter, m, is between zero and one. Then, a descending Landen (Gauss) transform is applied until the parameter is small. The small parameter approximation is then applied.

Example

In this example, $\mathrm{dn}(1.5 + 0.3i, 0.5)$ is computed and printed.

```
C                                  Declare variables
      INTEGER    NOUT
      REAL       AM
      COMPLEX    CEJDN, VALUE, Z
      EXTERNAL   CEJDN, UMACH
C                                  Compute
      Z     = (1.5, 0.3)
      AM    = 0.5
      VALUE = CEJDN(Z, AM)
C                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, AM, VALUE
99999 FORMAT (' CEJDN((', F6.3, ',', F6.3, '), ', F6.3, ') = (',
     &     F6.3, ',', F6.3, ')')
      END
```

Output

```
CEJDN(( 1.500, 0.300), 0.500) = ( 0.714,-0.037)
```

Chapter 11: Probability Distribution Functions and Inverses

Routines

11.1. Discrete Random Variables: Distribution Functions and Probability Functions

Binomial distribution function .... BINDF 172
Binomial probability .... BINPR 173
Hypergeometric distribution function .... HYPDF 175
Hypergeometric probability .... HYPPR 177
Poisson distribution function .... POIDF 178
Poisson probability .... POIPR 180

11.2. Continuous Random Variables: Distribution Functions and Their Inverses

Kolmogorov-Smirnov one-sided statistic distribution function AKS1DF 181
Kolmogorov-Smirnov two-sided statistic distribution function AKS2DF 184
Normal (Gaussian) distribution function .... ANORDF 186
Inverse of the normal distribution function .... ANORIN 188
Beta distribution function .... BETDF 189
Inverse of the beta distribution function .... BETIN 191
Bivariate normal distribution function .... BNRDF 192
Chi-squared distribution function .... CHIDF 193
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Noncentral chi-squared distribution function .... CSNDF 197
F distribution function .... FDF 200
Inverse of the *F* distribution function .... FIN 201
Gamma distribution function .... GAMDF 203
Student's *t* distribution function .... TDF 205
Inverse of the Student's *t* distribution function .... TIN 207
Noncentral Student's *t* distribution function .... TNDF 208

11.3. **General Continuous Random Variables**
Distribution function given ordinates of density .... GCDF 210
Inverse of distribution function given ordinates of density ...... GCIN 212

Usage Notes

Definitions and discussions of the terms basic to this chapter can be found in Johnson and Kotz (1969, 1970a, 1970b). These are also good references for the specific distributions.

In order to keep the calling sequences simple, whenever possible, the routines in this chapter are written for standard forms of statistical distributions. Hence, the number of parameters for any given distribution may be fewer than the number often associated with the distribution. For example, while a gamma distribution is often characterized by two parameters (or even a third, "location"), there is only one parameter that is necessary, the "shape." The "scale" parameter can be used to scale the variable to the *standard* gamma distribution. For another example, the functions relating to the normal distribution, `ANORDF` (page 186) and `ANORIN` (page 188), are for a normal distribution with mean equal to zero and variance equal to one. For other means and variances, it is very easy for the user to standardize the variables by subtracting the mean and dividing by the square root of the variance.

The *distribution function* for the (real, single-valued) random variable X is the function F defined for all real x by

$$F(x) = \mathrm{Prob}(X \le x)$$

where Prob(·) denotes the probability of an event. The distribution function is often called the *cumulative distribution function* (CDF).

For distributions with finite ranges, such as the beta distribution, the CDF is 0 for values less than the left endpoint and 1 for values greater than the right endpoint. The routines in this chapter return the correct values for the distribution functions when values outside of the range of the random variable are input, but warning error conditions are set in these cases.

Discrete Random Variables

For discrete distributions, the function giving the probability that the random variable takes on specific values is called the *probability function*, defined by

$$p(x) = \mathrm{Prob}(X = x)$$

The "`PR`" routines in this chapter evaluate probability functions.

The CDF for a discrete random variable is

$$F(x) = \sum_{A} p(k)$$

where A is the set such that $k \le x$. The "DF" routines in this chapter evaluate cumulative distributions functions. Since the distribution function is a step function, its inverse does not exist uniquely.

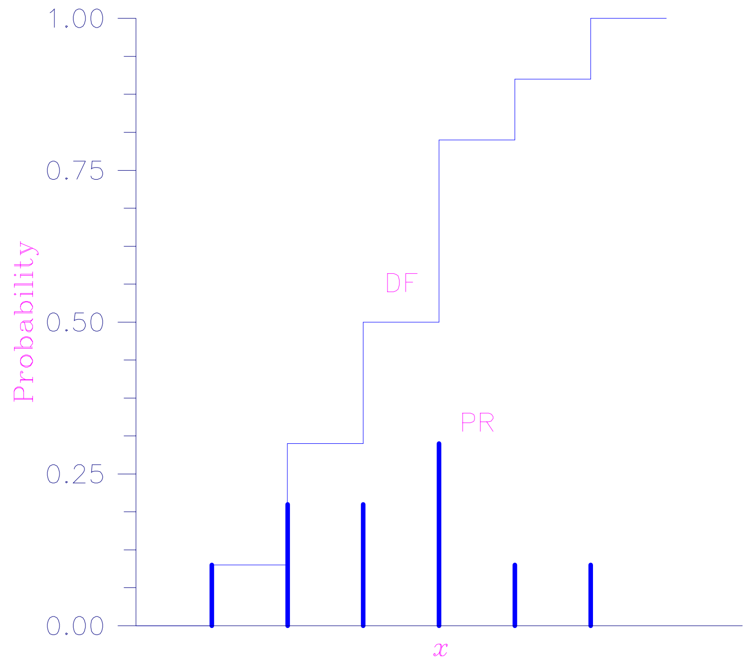

Figure 11-1 Discrete Random Variable

In the plot above, a routine like `BINPR` (page 173) in this chapter evaluates the individual probability, given X. A routine like `BINDF` (page 172) would evaluate the sum of the probabilities up to and including the probability at X.

Continuous Distributions

For continuous distributions, a probability function, as defined above, would not be useful because the probability of any given point is 0. For such distributions, the useful analog is the *probability density function* (PDF). The integral of the PDF is the probability over the interval; if the continuous random variable X has PDF f, then

$$\text{Prob}(a < X \le b) = \int_a^b f(x)\,dx$$

The relationship between the CDF and the PDF is

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

as shown below.

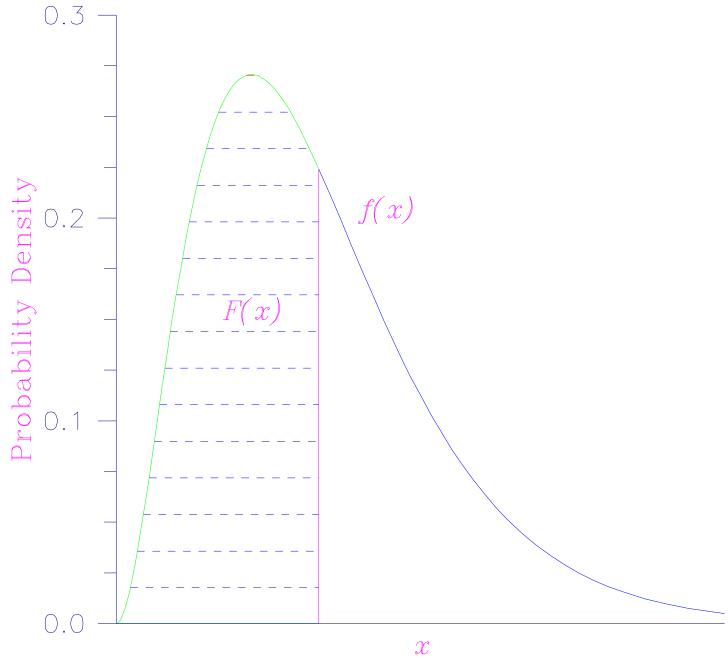

Figure 11-2 Probability Density Function

The “`DF`” routines for continuous distributions in this chapter evaluate cumulative distribution functions, just as the ones for discrete distributions.

For (absolutely) continuous distributions, the value of $F(x)$ uniquely determines x within the support of the distribution. The “`IN`” routines in this chapter compute the inverses of the distribution functions; that is, given $F(x)$ (called “`P`” for “probability”), a routine like `BETIN` (page 191) computes x. The inverses are defined only over the open interval (0, 1).

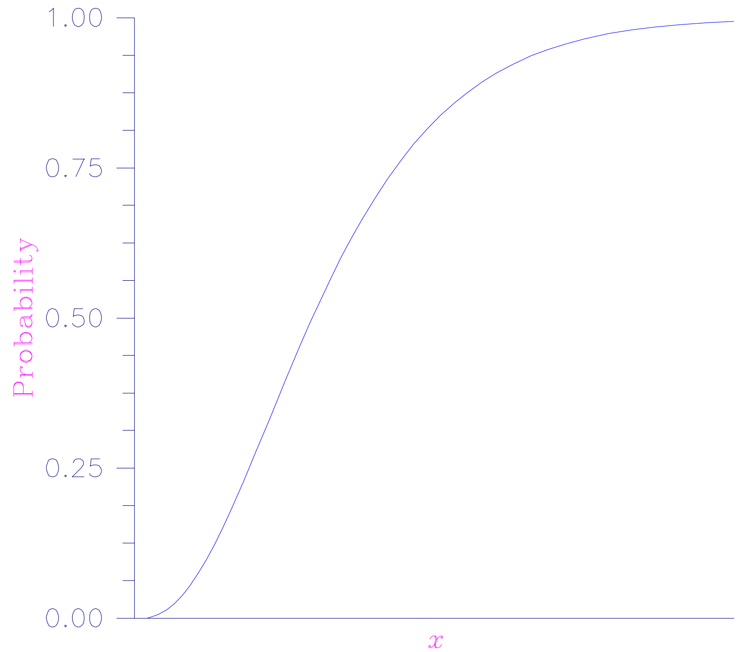

Figure 11-3   Cumulative Probability Distribution Function

There are two routines in this chapter that deal with general continuous distribution functions. The routine `GCDF` (page 210) computes a distribution function using values of the density function, and the routine `GCIN` (page 212) computes the inverse. These two routines may be useful when the user has an estimate of a probability density.

Additional Comments

Whenever a probability close to 1.0 results from a call to a distribution function or is to be input to an inverse function, it is often impossible to achieve good accuracy because of the nature of the representation of numeric values. In this case, it may be better to work with the complementary distribution function (one minus the distribution function). If the distribution is symmetric about some point (as the normal distribution, for example) or is reflective about some point (as the beta distribution, for example), the complementary distribution function has a simple relationship with the distribution function. For example, to evaluate the standard normal distribution at 4.0, using `ANORIN` (page 188) directly, the result to six places is 0.999968. Only two of those digits are really useful, however. A more useful result may be 1.000000 minus this value, which can be obtained to six significant figures as 3.16713E–05 by evaluating `ANORIN` at –4.0. For the normal distribution, the two values are related by $\Phi(x) = 1 - \Phi(-x)$, where $\Phi(\cdot)$ is the normal distribution function. Another example is the beta distribution with parameters 2 and 10. This distribution is skewed to the right; so evaluating BETDF at 0.7, we obtain 0.999953. A more precise result is obtained by evaluating BETDF with parameters 10 and 2 at 0.3. This yields 4.72392E−5. (In both of these examples, it is wise not to trust the last digit.)

Many of the algorithms used by routines in this chapter are discussed by Abramowitz and Stegun (1964). The algorithms make use of various expansions and recursive relationships, and often use different methods in different regions.

Cumulative distribution functions are defined for all real arguments; however, if the input to one of the distribution functions in this chapter is outside the range of the random variable, an error of Type 1 is issued, and the output is set to zero or one, as appropriate. A Type 1 error is of lowest severity, a "note;" and, by default, no printing or stopping of the program occurs. The other common errors that occur in the routines of this chapter are Type 2, "alert," for a function value being set to zero due to underflow; Type 3, "warning," for considerable loss of accuracy in the result returned; and Type 5, "terminal," for incorrect and/ or inconsistent input, complete loss of accuracy in the result returned, or inability to represent the result (because of overflow). When a Type 5 error occurs, the result is set to NaN (not a number, also used as a missing value code, obtained by IMSL routine AMACH(6) (page 240)). (See the section "User Errors" in the Reference Material.)

BINDF/DBINDF (Single/Double precision)

Evaluate the binomial distribution function.

Usage

```
BINDF(K, N, P)
```

Arguments

K — Argument for which the binomial distribution function is to be evaluated. (Input)

N — Number of Bernoulli trials. (Input)

P — Probability of success on each trial. (Input)

BINDF — Function value, the probability that a binomial random variable takes a value less than or equal to K. (Output)
BINDF is the probability that K or fewer successes occur in N independent Bernoulli trials, each of which has a P probability of success.

Comments

Informational errors

Type	Code	
1	3	The input argument, K, is less than zero.

1	4	The input argument, `K`, is greater than the number of Bernoulli trials, `N`.

Algorithm

Function `BINDF` evaluates the distribution function of a binomial random variable with parameters *n* and *p*. It does this by summing probabilities of the random variable taking on the specific values in its range. These probabilities are computed by the recursive relationship

$$\Pr(X = j) = \frac{(n+1-j)p}{j(1-p)} \Pr(X = j-1)$$

To avoid the possibility of underflow, the probabilities are computed forward from 0, if *k* is not greater than *n* times *p*, and are computed backward from *n*, otherwise. The smallest positive machine number, ε, is used as the starting value for summing the probabilities, which are rescaled by $(1-p)^n\varepsilon$ if forward computation is performed and by $p^n\varepsilon$ if backward computation is done.

For the special case of $p = 0$, `BINDF` is set to 1; and for the case $p = 1$, `BINDF` is set to 1 if $k = n$ and to 0 otherwise.

Example

Suppose *X* is a binomial random variable with $n = 5$ and $p = 0.95$. In this example, we find the probability that *X* is less than or equal to 3.

```
      INTEGER    K, N, NOUT
      REAL       BINDF, P, PR
      EXTERNAL   BINDF, UMACH
C
      CALL UMACH (2, NOUT)
      K  = 3
      N  = 5
      P  = 0.95
      PR = BINDF(K,N,P)
      WRITE (NOUT,99999) PR
99999 FORMAT (' The probability that X is less than or equal to 3 is '
     &       , F6.4)
      END
```

Output

```
The probability that X is less than or equal to 3 is 0.0226
```

BINPR/DBINPR (Single/Double precision)

Evaluate the binomial probability function.

Usage

`BINPR(K, N, P)`

Arguments

K — Argument for which the binomial probability function is to be evaluated. (Input)

N — Number of Bernoulli trials. (Input)

P — Probability of success on each trial. (Input)

BINPR — Function value, the probability that a binomial random variable takes a value equal to `K`. (Output)

Comments

Informational errors

Type	Code	
1	3	The input argument, `K`, is less than zero.
1	4	The input argument, `K`, is greater than the number of Bernoulli trials, `N`.

Algorithm

The function `BINPR` evaluates the probability that a binomial random variable with parameters *n* and *p* takes on the value *k*. It does this by computing probabilities of the random variable taking on the values in its range less than (or the values greater than) *k*. These probabilities are computed by the recursive relationship

$$\Pr(X = j) = \frac{(n+1-j)p}{j(1-p)}\Pr(X = j-1)$$

To avoid the possibility of underflow, the probabilities are computed forward from 0, if *k* is not greater than *n* times *p*, and are computed backward from *n*, otherwise. The smallest positive machine number, ε, is used as the starting value for computing the probabilities, which are rescaled by $(1-p)^n\varepsilon$ if forward computation is performed and by $p^n\varepsilon$ if backward computation is done.

For the special case of $p = 0$, `BINPR` is set to 0 if *k* is greater than 0 and to 1 otherwise; and for the case $p = 1$, `BINPR` is set to 0 if *k* is less than *n* and to 1 otherwise.

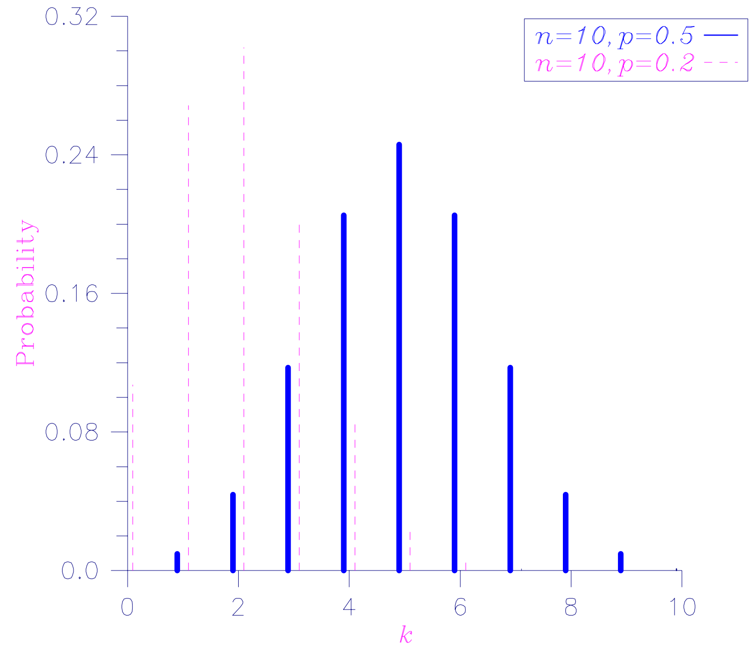

Figure 11-4 Binomial Probability Function

Example

Suppose X is a binomial random variable with $n = 5$ and $p = 0.95$. In this example, we find the probability that X is equal to 3.

```
      INTEGER    K, N, NOUT
      REAL       BINPR, P, PR
      EXTERNAL   BINPR, UMACH
C
      CALL UMACH (2, NOUT)
      K  = 3
      N  = 5
      P  = 0.95
      PR = BINPR(K,N,P)
      WRITE (NOUT,99999) PR
99999 FORMAT (' The probability that X is equal to 3 is ', F6.4)
      END
```

Output

```
The probability that X is equal to 3 is 0.0214
```

HYPDF/DHYPDF (Single/Double precision)

Evaluate the hypergeometric distribution function.

Usage

`HYPDF(K, N, M, L)`

Arguments

K — Argument for which the hypergeometric distribution function is to be evaluated. (Input)

N — Sample size. (Input)
`N` must be greater than zero and greater than or equal to `K`.

M — Number of defectives in the lot. (Input)

L — Lot size. (Input)
`L` must be greater than or equal to `N` and `M`.

HYPDF — Function value, the probability that a hypergeometric random variable takes a value less than or equal to `K`. (Output)
`HYPDF` is the probability that `K` or fewer defectives occur in a sample of size `N` drawn from a lot of size `L` that contains `M` defectives.

Comments

Informational errors

Type	Code	
1	5	The input argument, `K`, is less than zero.
1	6	The input argument, `K`, is greater than the sample size.

Algorithm

The function `HYPDF` evaluates the distribution function of a hypergeometric random variable with parameters *n*, *l*, and *m*. The hypergeometric random variable *X* can be thought of as the number of items of a given type in a random sample of size *n* that is drawn without replacement from a population of size *l* containing *m* items of this type. The probability function is

$$\Pr(X = j) = \frac{\binom{m}{j}\binom{l-m}{n-j}}{\binom{l}{n}} \quad \text{for } j = i,\ i+1,\ i+2,\ \ldots,\ \min(n, m)$$

where $i = \max(0, n - l + m)$.

If *k* is greater than or equal to *i* and less than or equal to min(*n*, *m*), `HYPDF` sums the terms in this expression for *j* going from *i* up to *k*. Otherwise, `HYPDF` returns 0 or 1, as appropriate. So, as to avoid rounding in the accumulation, `HYPDF` performs the summation differently depending on whether or not *k* is greater than the mode of the distribution, which is the greatest integer in $(m + 1)(n + 1)/(l + 2)$.

Example

Suppose X is a hypergeometric random variable with $n = 100$, $l = 1000$, and $m = 70$. In this example, we evaluate the distribution function at 7.

```
      INTEGER    K, L, M, N, NOUT
      REAL       DF, HYPDF
      EXTERNAL   HYPDF, UMACH
C
      CALL UMACH (2, NOUT)
      K  = 7
      N  = 100
      L  = 1000
      M  = 70
      DF = HYPDF(K,N,M,L)
      WRITE (NOUT,99999) DF
99999 FORMAT (' The probability that X is less than or equal to 7 is '
     &       , F6.4)
      END
```

Output

```
The probability that X is less than or equal to 7 is 0.5995
```

HYPPR/DHYPPR (Single/Double precision)

Evaluate the hypergeometric probability function.

Usage

`HYPPR(K, N, M, L)`

Arguments

K — Argument for which the hypergeometric probability function is to be evaluated. (Input)

N — Sample size. (Input)
`N` must be greater than zero and greater than or equal to `K`.

M — Number of defectives in the lot. (Input)

L — Lot size. (Input)
`L` must be greater than or equal to `N` and `M`.

HYPPR **—** Function value, the probability that a hypergeometric random variable takes a value equal to `K`. (Output)
`HYPPR` is the probability that exactly `K` defectives occur in a sample of size `N` drawn from a lot of size `L` that contains `M` defectives.

Comments

Informational errors

Type	Code	
1	5	The input argument, `K`, is less than zero.

1	6	The input argument, K, is greater than the sample size.

Algorithm

The function HYPPR evaluates the probability function of a hypergeometric random variable with parameters n, l, and m. The hypergeometric random variable X can be thought of as the number of items of a given type in a random sample of size n that is drawn without replacement from a population of size l containing m items of this type. The probability function is

$$\Pr(X = k) = \frac{\binom{m}{k}\binom{l-m}{n-k}}{\binom{l}{n}} \quad \text{for } k = i,\ i+1,\ i+2,\ \ldots \min(n, m)$$

where $i = \max(0, n - l + m)$.

HYPPR evaluates the expression using log gamma functions.

Example

Suppose X is a hypergeometric random variable with $n = 100$, $l = 1000$, and $m = 70$. In this example, we evaluate the probability function at 7.

```
      INTEGER    K, L, M, N, NOUT
      REAL       HYPPR, PR
      EXTERNAL   HYPPR, UMACH
C
      CALL UMACH (2, NOUT)
      K  = 7
      N  = 100
      L  = 1000
      M  = 70
      PR = HYPPR(K,N,M,L)
      WRITE (NOUT,99999) PR
99999 FORMAT (' The probability that X is equal to 7 is ', F6.4)
      END
```

Output

```
The probability that X is equal to 7 is 0.1628
```

POIDF/DPOIDF (Single/Double precision)

Evaluate the Poisson distribution function.

Usage

POIDF(K, THETA)

Arguments

K — Argument for which the Poisson distribution function is to be evaluated. (Input)

THETA — Mean of the Poisson distribution. (Input)
`THETA` must be positive.

POIDF — Function value, the probability that a Poisson random variable takes a value less than or equal to `K`. (Output)

Comments

Informational error

Type	Code	
1	1	The input argument, `K`, is less than zero.

Algorithm

The function `POIDF` evaluates the distribution function of a Poisson random variable with parameter `THETA`. `THETA`, which is the mean of the Poisson random variable, must be positive. The probability function (with θ = `THETA`) is

$$f(x) = e^{-\theta}\,\theta^{x}/x!,\ \text{for } x = 0, 1, 2, \ldots$$

The individual terms are calculated from the tails of the distribution to the mode of the distribution and summed. `POIDF` uses the recursive relationship

$$f(x+1) = f(x)\theta/(x+1),\ \text{for } x = 0, 1, 2, \ldots, k-1$$

with $f(0) = e^{-\theta}$.

Example

Suppose X is a Poisson random variable with θ = 10. In this example, we evaluate the distribution function at 7.

```
      INTEGER    K, NOUT
      REAL       DF, POIDF, THETA
      EXTERNAL   POIDF, UMACH
C
      CALL UMACH (2, NOUT)
      K     = 7
      THETA = 10.0
      DF    = POIDF(K,THETA)
      WRITE (NOUT,99999) DF
99999 FORMAT (' The probability that X is less than or equal to ',
     &       '7 is ', F6.4)
      END
```

Output

```
The probability that X is less than or equal to 7 is 0.2202
```

POIPR/DPOIPR (Single/Double precision)

Evaluate the Poisson probability function.

Usage

`POIPR(K, THETA)`

Arguments

K — Argument for which the Poisson distribution function is to be evaluated. (Input)

THETA — Mean of the Poisson distribution. (Input)
`THETA` must be positive.

POIPR — Function value, the probability that a Poisson random variable takes a value equal to `K`. (Output)

Comments

Informational error

Type	Code	
1	1	The input argument, `K`, is less than zero.

Algorithm

The function `POIPR` evaluates the probability function of a Poisson random variable with parameter `THETA`. `THETA`, which is the mean of the Poisson random variable, must be positive. The probability function (with θ = `THETA`) is

$$f(k) = e^{-\theta}\theta^{k}/k!, \text{ for } k = 0, 1, 2, \ldots$$

`POIPR` evaluates this function directly, taking logarithms and using the log gamma function.

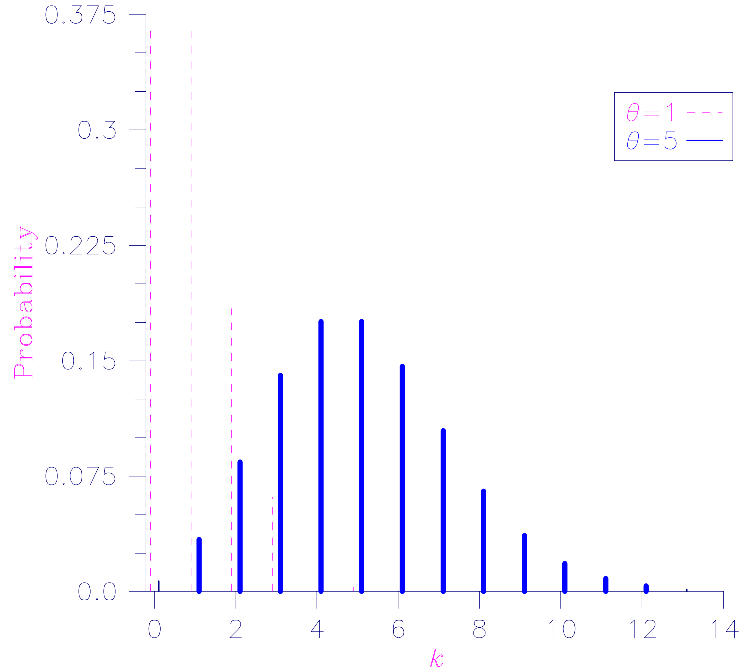

Figure 11-5   Poisson Probability Function

Example

Suppose X is a Poisson random variable with θ = 10. In this example, we evaluate the probability function at 7.

```
      INTEGER     K, NOUT
      REAL        POIPR, PR, THETA
      EXTERNAL    POIPR, UMACH
C
      CALL UMACH (2, NOUT)
      K     = 7
      THETA = 10.0
      PR    = POIPR(K,THETA)
      WRITE (NOUT,99999) PR
99999 FORMAT (' The probability that X is equal to 7 is ', F6.4)
      END
```

Output

```
The probability that X is equal to 7 is 0.0901
```

AKS1DF/DKS1DF (Single/Double precision)

Evaluate the distribution function of the one-sided Kolmogorov-Smirnov goodness of fit D^+ or D^- test statistic based on continuous data for one sample.

Usage

`AKS1DF(NOBS, D)`

Arguments

NOBS — The total number of observations in the sample. (Input)

D — The D^+ or D^- test statistic. (Input)
`D` is the maximum positive difference of the empirical cumulative distribution function (CDF) minus the hypothetical CDF or the maximum positive difference of the hypothetical CDF minus the empirical CDF.

AKS1DF — The probability of a smaller `D`. (Output)

Comments

1. Automatic workspace usage is

   `AKS1DF` 3 * (`NOBS` + 1) units, or
   `DKS1DF` 6 * (`NOBS` + 1) units.

   Workspace may be explicitly provided, if desired, by use of `AK21DF`/`DK21DF`. The reference is

   `AK2DF(NOBS, D, WK)`

   The additional argument is

   WK — Work vector of length 3 * `NOBS` + 3 if `NOBS` ≤ 80. `WK` is not used if `NOBS` is greater than 80.

2. Informational errors

Type	Code	
1	2	Since the *D* test statistic is less than zero, the distribution function is zero at *D*.
1	3	Since the *D* test statistic is greater than one, the distribution function is one at *D*.

3. If `NOBS` ≤ 80, then exact one-sided probabilities are computed. In this case, on the order of $\texttt{NOBS}^2$ operations are required. For `NOBS` > 80, approximate one-sided probabilities are computed. These approximate probabilities require very few computations.

4. An approximate two-sided probability for the `D` = max (D^+, D^-) statistic can be computed as twice the `AKS1DF` probability for `D` (minus one, if the probability from `AKS1DF` is greater than 0.5).

Algorithm

Routine `AKS1DF` computes the cumulative distribution function (CDF) for the one-sided Kolmogorov-Smirnov one-sample D^+ or D^- statistic when the theoretical CDF is strictly continuous. Exact probabilities are computed according to a method given by Conover (1980, page 350) for sample sizes of 80 or less. For sample sizes greater than 80, the asympotic methods discussed by Conover are used.

Let $F(x)$ denote the theoretical distribution function, and let $S_n(x)$ denote the empirical distribution function obtained from a sample of size NOBS. Then, the D^+ statistic is computed as

$$D^+ = \sup_x \left[F(x) - S_n(x) \right]$$

while the one-sided D^- statistic is computed as

$$D^- = \sup_x \left[S_n(x) - F(x) \right]$$

Programming Notes

Routine AKS1DF requires on the order of NOBS^2 operations to compute the exact probabilities, where an operation consists of taking ten or so logarithms. Because so much computation is occurring within each “operation,” AKS1DF is much slower than its two-sample counterpart, IMSL function AKS2DF (page 184).

Example

In this example, the exact one-sided probabilities for the tabled values of D^+ or D^-, given, for example, in Conover (1980, page 462), are computed. Tabled values at the 10% level of significance are used as input to AKS1DF for sample sizes of 5 to 50 in increments of 5. The last two tabled values are obtained using the asymptotic critical values of

$$1.07 / \sqrt{\text{NOBS}}$$

The resulting probabilities should all be close to 0.90.

```
      INTEGER    I, NOBS, NOUT
      REAL       AKS1DF, D(10)
      EXTERNAL   AKS1DF, UMACH
C
      DATA D/0.447, 0.323, 0.266, 0.232, 0.208, 0.190, 0.177, 0.165,
     &     0.160, 0.151/
C
      CALL UMACH (2, NOUT)
C
      DO 10  I=1, 10
         NOBS = 5*I
C
         WRITE (NOUT,99999) D(I), NOBS, AKS1DF(NOBS,D(I))
C
99999    FORMAT (' One-sided Probability for D = ', F8.3, ' with NOBS '
     &          , '= ', I2, ' is ', F8.4)
   10 CONTINUE
      END
```

Output

```
One-sided Probability for D =    0.447 with NOBS =  5 is   0.9000
One-sided Probability for D =    0.323 with NOBS = 10 is   0.9006
One-sided Probability for D =    0.266 with NOBS = 15 is   0.9002
One-sided Probability for D =    0.232 with NOBS = 20 is   0.9009
One-sided Probability for D =    0.208 with NOBS = 25 is   0.9002
One-sided Probability for D =    0.190 with NOBS = 30 is   0.8992
One-sided Probability for D =    0.177 with NOBS = 35 is   0.9011
One-sided Probability for D =    0.165 with NOBS = 40 is   0.8987
One-sided Probability for D =    0.160 with NOBS = 45 is   0.9105
One-sided Probability for D =    0.151 with NOBS = 50 is   0.9077
```

AKS2DF/DKS2DF (Single/Double precision)

Evaluate the distribution function of the Kolmogorov-Smirnov goodness of fit *D* test statistic based on continuous data for two samples.

Usage

`AKS2DF(NOBSX, NOBSY, D)`

Arguments

NOBSX — The total number of observations in the first sample. (Input)

NOBSY — The total number of observations in the second sample. (Input)

D — The *D* test statistic. (Input)
`D` is the maximum absolute difference between empirical cumulative distribution functions (CDFs) of the two samples.

AKS2DF — The probability of a smaller `D`. (Output)

Comments

1. Automatic workspace usage is

`AKS2DF`	max(`NOBSX`, `NOBSY`) + 1 units, or
`DKS2DF`	2 * max(`NOBSX`, `NOBSY`) + 1 units.

   Workspace may be explicitly provided, if desired, by use of `AK22DF/DK22DF`. The reference is

   `AK22DF(NOBSX, NOBSY, D, WK)`

   The additional argument is

   WK — Work vector of length max(`NOBSX`, `NOBSY`) + 1.

2. Informational errors

Type	Code	
1	2	Since the *D* test statistic is less than zero, then the distribution function is zero at `D`.

1	3	Since the *D* test statistic is greater than one, then the distribution function is one at D.

Algorithm

Function AKS2DF computes the cumulative distribution function (CDF) for the two-sided Kolmogorov-Smirnov two-sample *D* statistic when the theoretical CDF is strictly continuous. Exact probabilities are computed according to a method given by Kim and Jennrich (1973). Approximate asymptotic probabilities are computed according to methods also given in this reference.

Let $F_n(x)$ and $G_m(x)$ denote the empirical distribution functions for the two samples, based on n = NOBSX and m = NOBSY observations. Then, the *D* statistic is computed as

$$D = \sup_x \left| F_n(x) - G_m(x) \right|$$

Programming Notes

Function AKS2DF requires on the order of NOBSX * NOBSY operations to compute the exact probabilities, where an operation consists of an addition and a multiplication. For NOBSX * NOBSY less than 10000, the exact probability is computed. If this is not the case, then the Smirnov approximation discussed by Kim and Jennrich is used if the minimum of NOBSX and NOBSY is greater than ten percent of the maximum of NOBSX and NOBSY, or if the minimum is greater than 80. Otherwise, the Kolmogorov approximation discussed by Kim and Jennrich is used.

Example

Function AKS2DF is used to compute the probability of a smaller *D* statistic for a variety of sample sizes using values close to the 0.95 probability value.

```
      INTEGER    I, NOBSX(10), NOBSY(10), NOUT
      REAL       AKS2DF, D(10)
      EXTERNAL   AKS2DF, UMACH
C
      DATA NOBSX/5, 20, 40, 70, 110, 200, 200, 200, 100, 100/
      DATA NOBSY/10, 10, 10, 10, 10, 20, 40, 60, 80, 100/
      DATA D/0.7, 0.55, 0.475, 0.4429, 0.4029, 0.2861, 0.2113, 0.1796,
     &     0.18, 0.18/
C
      CALL UMACH (2, NOUT)
C
      DO 10  I=1, 10
C
         WRITE (NOUT,99999) D(I), NOBSX(I), NOBSY(I),
     &                     AKS2DF(NOBSX(I),NOBSY(I),D(I))
C
99999    FORMAT (' Probability for D = ', F5.3, ' with NOBSX = ', I3,
     &          ' and NOBSY = ', I3, ' is ', F9.6, '.')
   10 CONTINUE
```

```
      END
```

Output

```
Probability for D = 0.700 with NOBSX =   5 and NOBSY =  10 is  0.980686.
Probability for D = 0.550 with NOBSX =  20 and NOBSY =  10 is  0.987553.
Probability for D = 0.475 with NOBSX =  40 and NOBSY =  10 is  0.972423.
Probability for D = 0.443 with NOBSX =  70 and NOBSY =  10 is  0.961646.
Probability for D = 0.403 with NOBSX = 110 and NOBSY =  10 is  0.928667.
Probability for D = 0.286 with NOBSX = 200 and NOBSY =  20 is  0.921126.
Probability for D = 0.211 with NOBSX = 200 and NOBSY =  40 is  0.917110.
Probability for D = 0.180 with NOBSX = 200 and NOBSY =  60 is  0.914520.
Probability for D = 0.180 with NOBSX = 100 and NOBSY =  80 is  0.908185.
Probability for D = 0.180 with NOBSX = 100 and NOBSY = 100 is  0.946098.
```

ANORDF/DNORDF (Single/Double precision)

Evaluate the standard normal (Gaussian) distribution function.

Usage

`ANORDF(X)`

Arguments

***X* —** Argument for which the normal distribution function is to be evaluated. (Input)

***ANORDF* —** Function value, the probability that a normal random variable takes a value less than or equal to `X`. (Output)

Algorithm

Function `ANORDF` evaluates the distribution function, Φ, of a standard normal (Gaussian) random variable, that is,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

The value of the distribution function at the point *x* is the probability that the random variable takes a value less than or equal to *x*.

The standard normal distribution (for which `ANORDF` is the distribution function) has mean of 0 and variance of 1. The probability that a normal random variable with mean μ and variance σ^2 is less than *y* is given by `ANORDF` evaluated at $(y - \mu)/\sigma$.

Φ(*x*) is evaluated by use of the complementary error function, erfc. (See `ERFC`, page 71) The relationship is:

$$\Phi(x) = \operatorname{erfc}\left(-x / \sqrt{2.0}\right) / 2$$

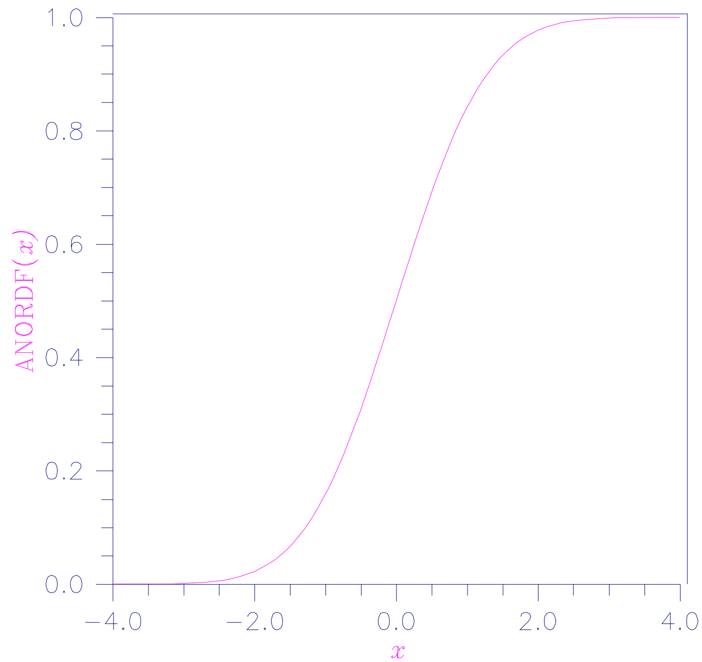

Figure 11-6   Standard Normal Distribution Function

Example

Suppose X is a normal random variable with mean 100 and variance 225. In this example, we find the probability that X is less than 90, and the probability that X is between 105 and 110.

```
      INTEGER    NOUT
      REAL       ANORDF, P, X1, X2
      EXTERNAL   ANORDF, UMACH
C
      CALL UMACH (2, NOUT)
      X1 = (90.0-100.0)/15.0
      P  = ANORDF(X1)
      WRITE (NOUT,99998) P
99998 FORMAT (' The probability that X is less than 90 is ', F6.4)
      X1 = (105.0-100.0)/15.0
      X2 = (110.0-100.0)/15.0
      P  = ANORDF(X2) - ANORDF(X1)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that X is between 105 and 110 is ',
     &       F6.4)
      END
```

Output

```
The probability that X is less than 90 is 0.2525
The probability that X is between 105 and 110 is 0.1169
```

ANORIN/DNORIN (Single/Double precision)

Evaluate the inverse of the standard normal (Gaussian) distribution function.

Usage

`ANORIN(P)`

Arguments

P — Probability for which the inverse of the normal distribution function is to be evaluated. (Input)
`P` must be in the open interval (0.0, 1.0).

ANORIN — Function value. (Output)
The probability that a standard normal random variable takes a value less than or equal to `ANORIN` is `P`.

Algorithm

Function `ANORIN` evaluates the inverse of the distribution function, Φ, of a standard normal (Gaussian) random variable, that is, `ANORIN(P)` = $\Phi^{-1}(p)$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

The value of the distribution function at the point x is the probability that the random variable takes a value less than or equal to x. The standard normal distribution has a mean of 0 and a variance of 1.

References used to design this routine include Hart et al. (1968), Kinnucan and Kuki (1968), and Strecok (1968).

Example

In this example, we compute the point such that the probability is 0.9 that a standard normal random variable is less than or equal to this point.

```
      INTEGER    NOUT
      REAL       ANORIN, P, X
      EXTERNAL   ANORIN, UMACH
C
      CALL UMACH (2, NOUT)
      P = 0.9
      X = ANORIN(P)
      WRITE (NOUT,99999) X
99999 FORMAT (' The 90th percentile of a standard normal is ', F6.4)
      END
```

Output

```
The 90th percentile of a standard normal is 1.2816
```

BETDF/DBETDF (Single/Double precision)

Evaluate the beta probability distribution function.

Usage

`BETDF(X, PIN, QIN)`

Arguments

X — Argument for which the beta distribution function is to be evaluated. (Input)

PIN — First beta distribution parameter. (Input)
`PIN` must be positive.

QIN — Second beta distribution parameter. (Input)
`QIN` must be positive.

BETDF — Probability that a random variable from a beta distribution having parameters `PIN` and `QIN` will be less than or equal to `X`. (Output)

Comments

Informational errors

Type	Code	
1	1	Since the input argument `X` is less than or equal to zero, the distribution function is equal to zero at `X`.
1	2	Since the input argument `X` is greater than or equal to one, the distribution function is equal to one at `X`.

Algorithm

Function `BETDF` evaluates the distribution function of a beta random variable with parameters `PIN` and `QIN`. This function is sometimes called the *incomplete beta ratio* and, with p = `PIN` and q = `QIN`, is denoted by $I_x(p, q)$. It is given by

$$I_x(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \int_0^x t^{p-1}(1-t)^{q-1}\, dt$$

where $\Gamma(\cdot)$ is the gamma function. The value of the distribution function $I_x(p, q)$ is the probability that the random variable takes a value less than or equal to x.

The integral in the expression above is called the *incomplete beta function* and is denoted by $\beta_x(p, q)$. The constant in the expression is the reciprocal of the *beta function* (the incomplete function evaluated at one) and is denoted by $\beta(p, q)$.

Function `BETDF` uses the method of Bosten and Battiste (1974b).

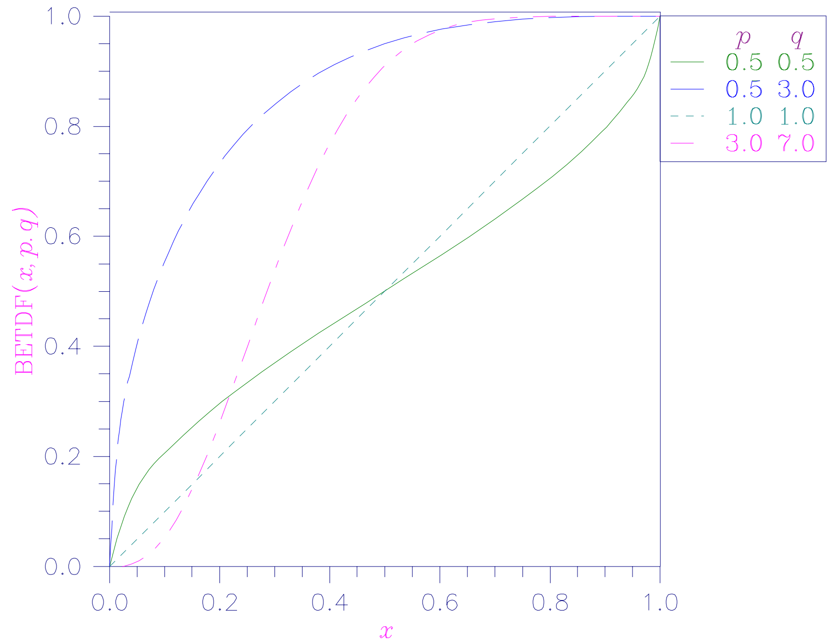

Figure 11-7   Beta Distribution Function

Example

Suppose *X* is a beta random variable with parameters 12 and 12. (*X* has a symmetric distribution.) In this example, we find the probability that *X* is less than 0.6 and the probability that *X* is between 0.5 and 0.6. (Since *X* is a symmetric beta random variable, the probability that it is less than 0.5 is 0.5.)

```
      INTEGER    NOUT
      REAL       BETDF, P, PIN, QIN, X
      EXTERNAL   BETDF, UMACH
C
      CALL UMACH (2, NOUT)
      PIN = 12.0
      QIN = 12.0
      X   = 0.6
      P   = BETDF(X,PIN,QIN)
      WRITE (NOUT,99998) P
99998 FORMAT (' The probability that X is less than 0.6 is ', F6.4)
      X = 0.5
      P = P - BETDF(X,PIN,QIN)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that X is between 0.5 and 0.6 is ',
     &       F6.4)
      END
```

Output

```
The probability that X is less than 0.6 is 0.8364
The probability that X is between 0.5 and 0.6 is 0.3364
```

BETIN/DBETIN (Single/Double precision)

Evaluate the inverse of the beta distribution function.

Usage

BETIN(P, PIN, QIN)

Arguments

P — Probability for which the inverse of the beta distribution function is to be evaluated. (Input)
P must be in the open interval (0.0, 1.0).

PIN — First beta distribution parameter. (Input)
PIN must be positive.

QIN — Second beta distribution parameter. (Input)
QIN must be positive.

BETIN — Function value. (Output)
The probability that a beta random variable takes a value less than or equal to BETIN is P.

Comments

Informational error

Type	Code	
3	1	The value for the inverse Beta distribution could not be found in 100 iterations. The best approximation is used.

Algorithm

The function BETIN evaluates the inverse distribution function of a beta random variable with parameters PIN and QIN, that is, with P = P, p = PIN, and q = QIN; it determines x (= BETIN(P, PIN, QIN)), such that

$$P = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \int_0^x t^{p-1}(1-t)^{q-1}\,dt$$

where $\Gamma(\cdot)$ is the gamma function. The probability that the random variable takes a value less than or equal to x is P.

Example

Suppose X is a beta random variable with parameters 12 and 12. (X has a symmetric distribution.) In this example, we find the value x_0 such that the probability that $X \le x_0$ is 0.9.

```
INTEGER    NOUT
REAL       BETIN, P, PIN, QIN, X
```

```
      EXTERNAL   BETIN, UMACH
C
      CALL UMACH (2, NOUT)
      PIN = 12.0
      QIN = 12.0
      P   = 0.9
      X   = BETIN(P,PIN,QIN)
      WRITE (NOUT,99999) X
99999 FORMAT (' X is less than ', F6.4, ' with probability 0.9.')
      END
```

Output

```
X is less than 0.6299 with probability 0.9.
```

BNRDF/DBNRDF (Single/Double precision)

Evaluate the bivariate normal distribution function.

Usage

`BNRDF(X, Y, RHO)`

Arguments

X — One argument for which the bivariate normal distribution function is to be evaluated. (Input)

Y — The other argument for which the bivariate normal distribution function is to be evaluated. (Input)

RHO — Correlation coefficient. (Input)

BNRDF — Function value, the probability that a bivariate normal random variable with correlation `RHO` takes a value less than or equal to `X` and less than or equal to `Y`. (Output)

Algorithm

Function `BNRDF` evaluates the distribution function F of a bivariate normal distribution with means of zero, variances of one, and correlation of `RHO`, that is, with ρ = `RHO`, and $|\rho| < 1$,

$$F(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left(-\frac{u^2 - 2\rho uv + v^2}{2\left(1-\rho^2\right)} \right) du\, dv$$

To determine the probability that $U \le u_0$ and $V \le v_0$, where $(U, V)^T$ is a bivariate normal random variable with mean $\mu = (\mu_U, \mu_V)^T$ and variance-covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_U^2 & \sigma_{UV} \\ \sigma_{UV} & \sigma_V^2 \end{pmatrix}$$

transform $(U, V)^T$ to a vector with zero means and unit variances. The input to BNRDF would be X = $(u_0 - \mu_U)/\sigma_U$, Y = $(v_0 - \mu_V) = \sigma_V$, and $\rho = \sigma_{UV}/(\sigma_U\sigma_V)$.

Function BNRDF uses the method of Owen (1962, 1965). For $|\rho| = 1$, the distribution function is computed based on the univariate statistic, $Z = \min(x, y)$, and on the normal distribution function ANORDF (page 186).

See Cooper (1968) for more information on the algorithm used.

Example

Suppose (X, Y) is a bivariate normal random variable with mean (0, 0) and variance-covariance matrix

$$\begin{pmatrix} 1.0 & 0.9 \\ 0.9 & 1.0 \end{pmatrix}$$

In this example, we find the probability that X is less than –2.0 and Y is less than 0.0.

```
      INTEGER    NOUT
      REAL       BNRDF, P, RHO, X, Y
      EXTERNAL   BNRDF, UMACH
C
      CALL UMACH (2, NOUT)
      X   = -2.0
      Y   = 0.0
      RHO = 0.9
      P   = BNRDF(X,Y,RHO)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that X is less than -2.0  and Y ',
     &       'is less than 0.0 is ', F6.4)
      END
```

Output

```
The probability that X is less than -2.0 and Y is less than 0.0 is 0.0228
```

CHIDF/DCHIDF (Single/Double precision)

Evaluate the chi-squared distribution function.

Usage

CHIDF(CHSQ, DF)

Arguments

CHSQ — Argument for which the chi-squared distribution function is to be evaluated. (Input)

DF — Number of degrees of freedom of the chi-squared distribution. (Input)
`DF` must be greater than or equal to 0.5.

CHIDF — Function value, the probability that a chi-squared random variable takes a value less than or equal to `CHSQ`. (Output)

Comments

Informational errors

Type	Code	
1	1	Since the input argument, `CHSQ`, is less than zero, the distribution function is zero at `CHSQ`.
2	3	The normal distribution is used for large degrees of freedom. However, it has produced underflow. Therefore, the probability, `CHIDF`, is set to zero.

Algorithm

Function `CHIDF` evaluates the distribution function, F, of a chi-squared random variable with `DF` degrees of freedom, that is, with ν = `DF`, and x = `CHSQ`,

$$F(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}\int_0^x e^{-t/2}t^{\nu/2-1}dt$$

where $\Gamma(\cdot)$ is the gamma function. The value of the distribution function at the point x is the probability that the random variable takes a value less than or equal to x.

For $\nu > 65$, `CHIDF` uses the Wilson-Hilferty approximation (Abramowitz and Stegun 1964, equation 26.4.17) to the normal distribution, and routine `ANORDF` (page 186) is used to evaluate the normal distribution function.

For $\nu \le 65$, `CHIDF` uses series expansions to evaluate the distribution function. If $x < \max(\nu/2, 26)$, `CHIDF` uses the series 6.5.29 in Abramowitz and Stegun (1964); otherwise, it uses the asymptotic expansion 6.5.32 in Abramowitz and Stegun.

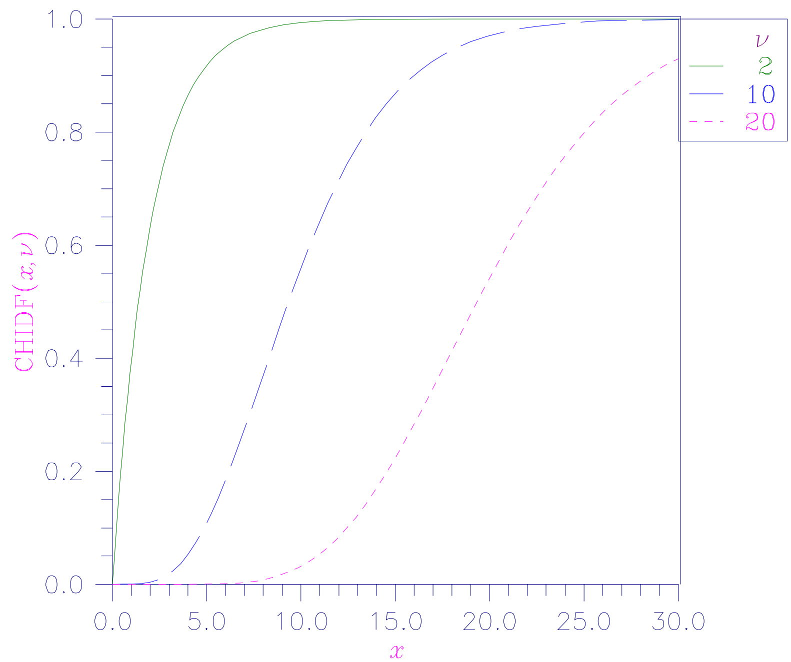

Figure 11-8 Chi-Squared Distribution Function

Example

Suppose X is a chi-squared random variable with 2 degrees of freedom. In this example, we find the probability that X is less than 0.15 and the probability that X is greater than 3.0.

```
      INTEGER    NOUT
      REAL       CHIDF, CHSQ, DF, P
      EXTERNAL   CHIDF, UMACH
C
      CALL UMACH (2, NOUT)
      DF   = 2.0
      CHSQ = 0.15
      P    = CHIDF(CHSQ,DF)
      WRITE (NOUT,99998) P
99998 FORMAT (' The probability that chi-squared with 2 df is less ',
     &        'than 0.15 is ', F6.4)
      CHSQ = 3.0
      P    = 1.0 - CHIDF(CHSQ,DF)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that chi-squared with 2 df is greater ',
     &        'than 3.0 is ', F6.4)
      END
```

Output

```
The probability that chi-squared with 2 df is less than 0.15 is 0.0723
The probability that chi-squared with 2 df is greater than 3.0 is 0.2231
```

CHIIN/DCHIIN (Single/Double precision)

Evaluate the inverse of the chi-squared distribution function.

Usage

```
CHIIN(P, DF)
```

Arguments

P — Probability for which the inverse of the chi-squared distribution function is to be evaluated. (Input)
P must be in the open interval (0.0, 1.0).

DF — Number of degrees of freedom of the chi-squared distribution. (Input)
DF must be greater than or equal to 0.5.

CHIIN — Function value. (Output)
The probability that a chi-squared random variable takes a value less than or equal to CHIIN is P.

Comments

Informational errors

Type	Code	
4	1	Over 100 iterations have occurred without convergence. Convergence is assumed.

Algorithm

Function CHIIN evaluates the inverse distribution function of a chi-squared random variable with DF degrees of freedom; that is, with P = P and ν = DF, it determines x (= CHIIN(P, DF)), such that

$$P = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}\int_0^x e^{-t/2}t^{\nu/2-1}dt$$

where $\Gamma(\cdot)$ is the gamma function. The probability that the random variable takes a value less than or equal to x is P.

For $\nu < 40$, CHIIN uses bisection (if $\nu \le 2$ or $P > 0.98$) or regula falsi to find the point at which the chi-squared distribution function is equal to P. The distribution function is evaluated using routine CHIDF (page 193).

For $40 \le \nu < 100$, a modified Wilson-Hilferty approximation (Abramowitz and Stegun 1964, equation 26.4.18) to the normal distribution is used, and routine ANORIN (page 188) is used to evaluate the inverse of the normal distribution function. For $\nu \ge 100$, the ordinary Wilson-Hilferty approximation (Abramowitz and Stegun 1964, equation 26.4.17) is used.

Example

In this example, we find the 99-th percentage points of a chi-squared random variable with 2 degrees of freedom and of one with 64 degrees of freedom.

```
      INTEGER    NOUT
      REAL       CHIIN, DF, P, X
      EXTERNAL   CHIIN, UMACH
C
      CALL UMACH (2, NOUT)
      P  = 0.99
      DF = 2.0
      X  = CHIIN(P,DF)
      WRITE (NOUT,99998) X
99998 FORMAT (' The 99-th percentage point of chi-squared with  2 df '
     &       , 'is ', F7.3)
      DF = 64.0
      X  = CHIIN(P,DF)
      WRITE (NOUT,99999) X
99999 FORMAT (' The 99-th percentage point of chi-squared with 64 df '
     &       , 'is ', F7.3)
      END
```

Output

```
The 99-th percentage point of chi-squared with 2 df is 9.210
The 99-th percentage point of chi-squared with 64 df is 93.217
```

CSNDF/DCSNDF (Single/Double precision)

Evaluate the noncentral chi-squared distribution function.

Usage

`CSNDF(CHSQ, DF, ALAM)`

Arguments

CHSQ — Argument for which the noncentral chi-squared distribution function is to be evaluated. (Input)

DF — Number of degrees of freedom of the noncentral chi-squared distribution. (Input)
`DF` must be greater than or equal to 0.5 and less than or equal to 200,000.

ALAM — The noncentrality parameter. (Input)
`ALAM` must be nonnegative, and `ALAM` + `DF` must be less than or equal to 200,000.

CSNDF — Function value, the probability that a noncentral chi-squared random variable takes a value less than or equal to `CHSQ`. (Output)

Comments

1. Informational errors

Type	Code	
1	1	Since the input argument, CHSQ, is less than or equal to zero, the distribution function is zero at CHSQ.
3	2	Convergence was not obtained. The best approximation to the probability is returned.

2. This subroutine sums terms of an infinite series of central chi-squared distribution functions weighted by Poisson terms. Summing terminates when either the current term is less than 10 * AMACH(4) times the current sum or when 1000 terms have been accumulated. In the latter case, a warning error is issued.

Algorithm

Function CSNDF evaluates the distribution function of a noncentral chi-squared random variable with DF degrees of freedom and noncentrality parameter ALAM; that is, with ν = DF, λ = ALAM, and x = CHSQ,

$$\mathrm{CSNDF}(x) = \sum_{i=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i}{i!} \int_0^x \frac{t^{(\nu+2i)/2-1} e^{-t/2}}{2^{(\nu+2i)/2}\,\Gamma\left(\frac{\nu+2i}{2}\right)} dt$$

where Γ(·) is the gamma function. This is a series of central chi-squared distribution functions with Poisson weights. The value of the distribution function at the point x is the probability that the random variable takes a value less than or equal to x.

The noncentral chi-squared random variable can be defined by the distribution function above, or alternatively and equivalently, as the sum of squares of independent normal random variables. If Y_i have independent normal distributions with means μ_i and variances equal to one and

$$X = \sum_{i=1}^{n} Y_i^2$$

then X has a noncentral chi-squared distribution with n degrees of freedom and noncentrality parameter equal to

$$\sum_{i=1}^{n} \mu_i^2$$

With a noncentrality parameter of zero, the noncentral chi-squared distribution is the same as the chi-squared distribution.

Function CSNDF determines the point at which the Poisson weight is greatest, and then sums forward and backward from that point, terminating when the additional terms are sufficiently small or when a maximum of 1000 terms have been accumulated. The recurrence relation 26.4.8 of Abramowitz and Stegun

(1964) is used to speed the evaluation of the central chi-squared distribution functions.

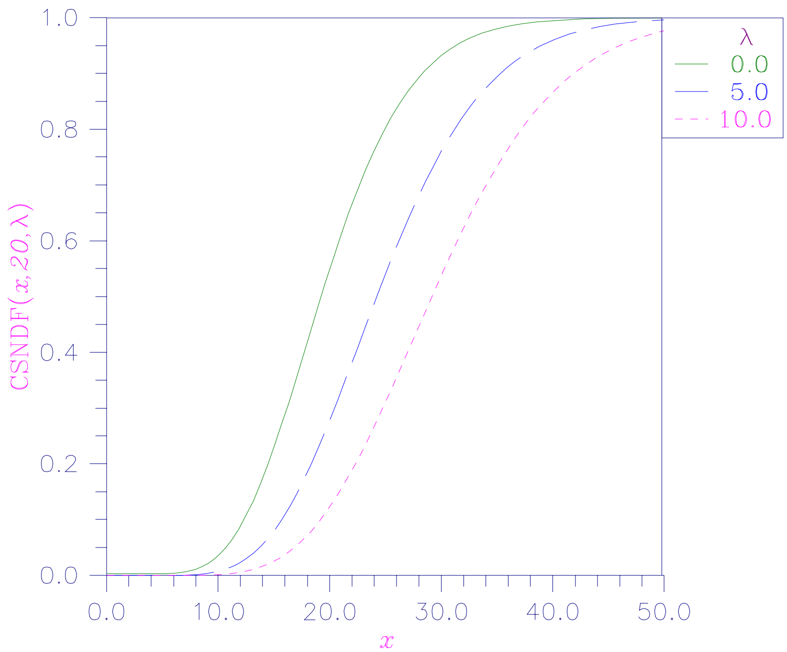

Figure 11-9   Noncentral Chi-squared Distribution Function

Example

In this example, `CSNDF` is used to compute the probability that a random variable that follows the noncentral chi-squared distribution with noncentrality parameter of 1 and with 2 degrees of freedom is less than or equal to 8.642.

```
      INTEGER    NOUT
      REAL       ALAM, CHSQ, CSNDF, DF, P
      EXTERNAL   CSNDF, UMACH
C
      CALL UMACH (2, NOUT)
      DF   = 2.0
      ALAM = 1.0
      CHSQ = 8.642
      P    = CSNDF(CHSQ,DF,ALAM)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that a noncentral chi-squared random',
     &       /, ' variable with 2 df and noncentrality 1.0 is less',
     &       /, ' than 8.642 is ', F5.3)
      END
```

Output

```
The probability that a noncentral chi-squared random
variable with 2 df and noncentrality 1.0 is less
than 8.642 is 0.950
```

FDF/DFDF (Single/Double precision)

Evaluate the F distribution function.

Usage

`FDF(F, DFN, DFD)`

Arguments

F — Argument for which the F distribution function is to be evaluated. (Input)

DFN — Numerator degrees of freedom. (Input)
`DFN` must be positive.

DFD — Denominator degrees of freedom. (Input)
`DFD` must be positive.

FDF — Function value, the probability that an F random variable takes a value less than or equal to the input `F`. (Output)

Comments

Informational error

Type	Code	
1	3	Since the input argument `F` is not positive, the distribution function is zero at `F`.

Algorithm

Function `FDF` evaluates the distribution function of a Snedecor's F random variable with `DFN` numerator degrees of freedom and `DFD` denominator degrees of freedom. The function is evaluated by making a transformation to a beta random variable and then using the routine `BETDF` (page 189). If X is an F variate with ν_1 and ν_2 degrees of freedom and $Y = \nu_1 X/(\nu_2 + \nu_1 X)$, then Y is a beta variate with parameters $p = \nu_1/2$ and $q = \nu_2/2$. The function `FDF` also uses a relationship between F random variables that can be expressed as follows:

```
FDF(X, DFN, DFD) = 1.0 - FDF(1.0/X, DFD, DFN)
```

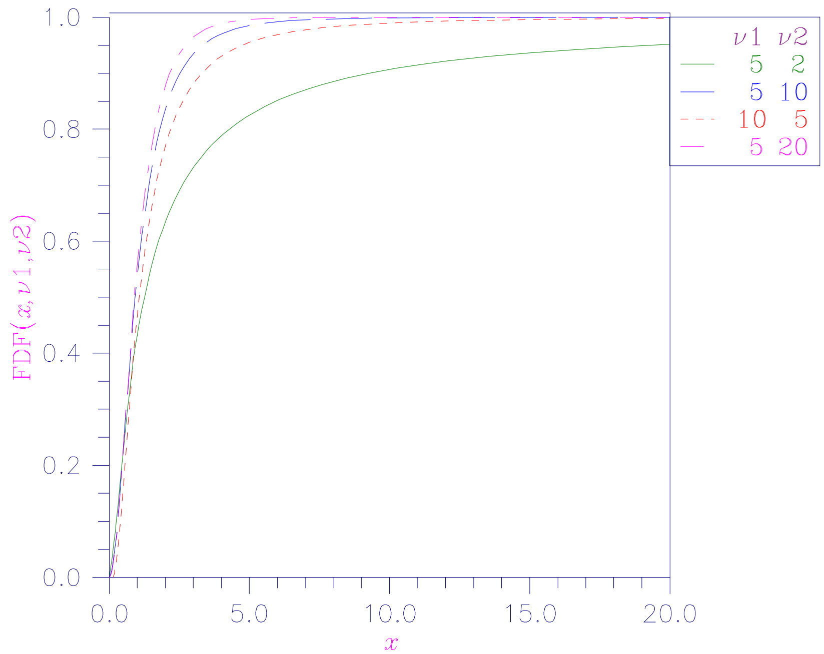

Figure 11-10 *F* Distribution Function

Example

In this example, we find the probability that an *F* random variable with one numerator and one denominator degree of freedom is greater than 648.

```
      INTEGER    NOUT
      REAL       DFD, DFN, F, FDF, P
      EXTERNAL   FDF, UMACH
C
      CALL UMACH (2, NOUT)
      F   = 648.0
      DFN = 1.0
      DFD = 1.0
      P   = 1.0 - FDF(F,DFN,DFD)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that an F(1,1) variate is greater ',
     &       'than 648 is ', F6.4)
      END
```

Output

```
The probability that an F(1,1) variate is greater than 648 is 0.0250
```

FIN/DFIN (Single/Double precision)

Evaluate the inverse of the *F* distribution function.

Usage

`FIN(P, DFN, DFD)`

Arguments

P — Probability for which the inverse of the *F* distribution function is to be evaluated. (Input)
`P` must be in the open interval (0.0, 1.0).

DFN — Numerator degrees of freedom. (Input)
`DFN` must be positive.

DFD — Denominator degrees of freedom. (Input)
`DFD` must be positive.

***FIN* — Function value.** (Output)
The probability that an *F* random variable takes a value less than or equal to `FIN` is `P`.

Comments

Informational error

Type	Code	
4	4	`FIN` is set to machine infinity since overflow would occur upon modifying the inverse value for the *F* distribution with the result obtained from the inverse `BETA` distribution.

Algorithm

Function `FIN` evaluates the inverse distribution function of a Snedecor's *F* random variable with `DFN` numerator degrees of freedom and `DFD` denominator degrees of freedom. The function is evaluated by making a transformation to a beta random variable and then using the routine `BETIN` (page 191). If X is an F variate with ν_1 and ν_2 degrees of freedom and $Y = \nu_1 X/(\nu_2 + \nu_1 X)$, then Y is a beta variate with parameters $p = \nu_1/2$ and $q = \nu_2/2$. If $P \leq 0.5$, `FIN` uses this relationship directly; otherwise, it also uses a relationship between *F* random variables that can be expressed as follows, using routine `FDF` (page 200), which is the *F* cumulative distribution function:

```
FDF(F, DFN, DFD) = 1.0 - FDF(1.0/F, DFD, DFN)
```

Example

In this example, we find the 99-th percentage point for an *F* random variable with 1 and 7 degrees of freedom.

```
      INTEGER    NOUT
      REAL       DFD, DFN, F, FIN, P
      EXTERNAL   FIN, UMACH
C
      CALL UMACH (2, NOUT)
      P   = 0.99
```

```
      DFN = 1.0
      DFD = 7.0
      F   = FIN(P,DFN,DFD)
      WRITE (NOUT,99999) F
99999 FORMAT (' The F(1,7) 0.01 critical value is ', F6.3)
      END
```

Output

```
The F(1,7) 0.01 critical value is 12.246
```

GAMDF/DGAMDF (Single/Double precision)

Evaluate the gamma distribution function.

Usage

`GAMDF(X, A)`

Arguments

X — Argument for which the gamma distribution function is to be evaluated. (Input)

A — The shape parameter of the gamma distribution. (Input)
This parameter must be positive.

GAMDF — Function value, the probability that a gamma random variable takes a value less than or equal to `X`. (Output)

Comments

Informational error

Type	Code	
1	2	Since the input argument `X` is less than zero, the distribution function is set to zero.

Algorithm

Function `GAMDF` evaluates the distribution function, F, of a gamma random variable with shape parameter a; that is,

$$F(x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt$$

where $\Gamma(\cdot)$ is the gamma function. (The gamma function is the integral from 0 to ∞ of the same integrand as above). The value of the distribution function at the point x is the probability that the random variable takes a value less than or equal to x.

The gamma distribution is often defined as a two-parameter distribution with a scale parameter b (which must be positive), or even as a three-parameter distribution in which the third parameter c is a location parameter. In the most general case, the probability density function over (c, ∞) is

$$f(t) = \frac{1}{b^a \Gamma(a)} e^{-(t-c)/b} (x-c)^{a-1}$$

If T is such a random variable with parameters a, b, and c, the probability that $T \leq t_0$ can be obtained from `GAMDF` by setting `X` = $(t_0 - c)/b$.

If `X` is less than a or if `X` is less than or equal to 1.0, `GAMDF` uses a series expansion. Otherwise, a continued fraction expansion is used. (See Abramowitz and Stegun, 1964.)

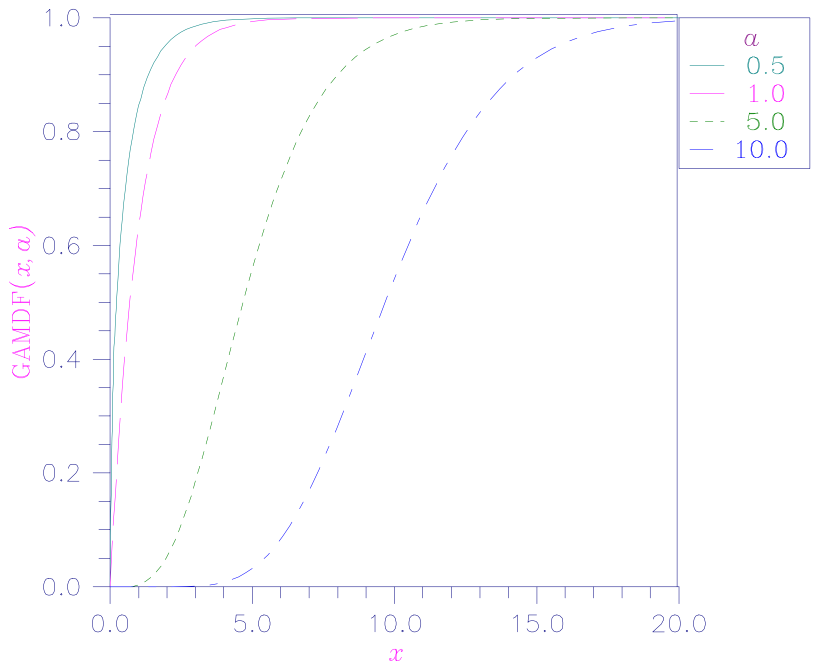

Figure 11-11 Gamma Distribution Function

Example

Suppose `X` is a gamma random variable with a shape parameter of 4. (In this case, it has an *Erlang distribution* since the shape parameter is an integer.) In this example, we find the probability that `X` is less than 0.5 and the probability that `X` is between 0.5 and 1.0.

```
      INTEGER    NOUT
      REAL       A, GAMDF, P, X
      EXTERNAL   GAMDF, UMACH
C
      CALL UMACH (2, NOUT)
      A = 4.0
```

```
      X = 0.5
      P = GAMDF(X,A)
      WRITE (NOUT,99998) P
99998 FORMAT (' The probability that X is less than 0.5 is ', F6.4)
      X = 1.0
      P = GAMDF(X,A) - P
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that X is between 0.5 and 1.0 is ',
     &       F6.4)
      END
```

Output

```
The probability that X is less than 0.5 is 0.0018
The probability that X is between 0.5 and 1.0 is 0.0172
```

TDF/DTDF (Single/Double precision)

Evaluate the Student's *t* distribution function.

Usage

`TDF(T, DF)`

Arguments

T — Argument for which the Student's *t* distribution function is to be evaluated. (Input)

DF — Degrees of freedom. (Input)
`DF` must be greater than or equal to 1.0.

TDF — Function value, the probability that a Student's *t* random variable takes a value less than or equal to the input `T`. (Output)

Algorithm

Function `TDF` evaluates the distribution function of a Student's *t* random variable with `DF` degrees of freedom. If the square of `T` is greater than or equal to `DF`, the relationship of a *t* to an *F* random variable (and subsequently, to a beta random variable) is exploited; and routine `BETDF` (page 189) is used. Otherwise, the method described by Hill (1970) is used. If `DF` is not an integer, if `DF` is greater than 19, or if `DF` is greater than 200, a Cornish-Fisher expansion is used to evaluate the distribution function. If `DF` is less than 20 and `ABS(T)` is less than 2.0, a trigonometric series (see Abramowitz and Stegun, 1964, equations 26.7.3 and 26.7.4, with some rearrangement) is used. For the remaining cases, a series given by Hill (1970) that converges well for large values of `T` is used.

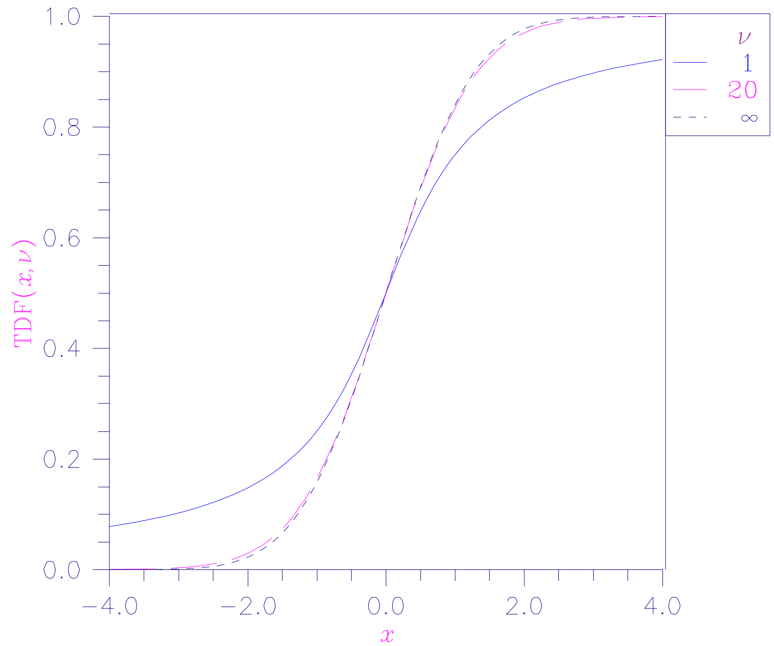

Figure 11-12 Student's *t* Distribution Function

Example

In this example, we find the probability that a *t* random variable with 6 degrees of freedom is greater in absolute value than 2.447. We use the fact that *t* is symmetric about 0.

```
      INTEGER    NOUT
      REAL       DF, P, T, TDF
      EXTERNAL   TDF, UMACH
C
      CALL UMACH (2, NOUT)
      T  = 2.447
      DF = 6.0
      P  = 2.0*TDF(-T,DF)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that a t(6) variate is greater ',
     &       'than 2.447 in', /, ' absolute value is ', F6.4)
      END
```

Output

```
The probability that a t(6) variate is greater than 2.447 in
absolute value is 0.0500
```

TIN/DTIN (Single/Double precision)

Evaluate the inverse of the Student's *t* distribution function.

Usage

`TIN(P, DF)`

Arguments

P — Probability for which the inverse of the Student's *t* distribution function is to be evaluated. (Input)
`P` must be in the open interval (0.0, 1.0).

DF — Degrees of freedom. (Input)
`DF` must be greater than or equal to 1.0.

TIN — Function value. (Output)
The probability that a Student's *t* random variable takes a value less than or equal to `TIN` is `P`.

Comments

Informational error

Type	Code	
4	3	`TIN` is set to machine infinity since overflow would occur upon modifying the inverse value for the *F* distribution with the result obtained from the inverse β distribution.

Algorithm

Function `TIN` evaluates the inverse distribution function of a Student's *t* random variable with `DF` degrees of freedom. Let ν = `DF`. If ν equals 1 or 2, the inverse can be obtained in closed form; if ν is between 1 and 2, the relationship of a *t* to a beta random variable is exploited and routine `BETIN` (page 191) is used to evaluate the inverse; otherwise the algorithm of Hill (1970) is used. For small values of ν greater than 2, Hill's algorithm inverts an integrated expansion in $1/(1 + t^2/\nu)$ of the *t* density. For larger values, an asymptotic inverse Cornish-Fisher type expansion about normal deviates is used.

Example

In this example, we find the 0.05 critical value for a two-sided *t* test with 6 degrees of freedom.

```
      INTEGER    NOUT
      REAL       DF, P, T, TIN
      EXTERNAL   TIN, UMACH
C
      CALL UMACH (2, NOUT)
      P  = 0.975
```

```
      DF = 6.0
      T  = TIN(P,DF)
      WRITE (NOUT,99999) T
99999 FORMAT (' The two-sided t(6) 0.05 critical value is ', F6.3)
      END
```

Output

```
The two-sided t(6) 0.05 critical value is 2.447
```

TNDF/DTNDF (Single/Double precision)

Evaluate the noncentral Student's *t* distribution function.

Usage

`TNDF(T, IDF, DELTA)`

Arguments

T — Argument for which the noncentral Student's *t* distribution function is to be evaluated. (Input)

IDF — Number of degrees of freedom of the noncentral Student's *t* distribution. (Input)
`IDF` must be positive.

DELTA — The noncentrality parameter. (Input)

TNDF — Function value, the probability that a noncentral Student's *t* random variable takes a value less than or equal to `T`. (Output)

Algorithm

Function `TNDF` evaluates the distribution function *F* of a noncentral *t* random variable with `IDF` degrees of freedom and noncentrality parameter `DELTA`; that is, with ν = `IDF`, δ = `DELTA`, and t_0 = `T`,

$$F(t_0) = \int_{-\infty}^{t_0} \frac{\nu^{\nu/2} e^{-\delta^2/2}}{\sqrt{\pi}\,\Gamma(\nu/2)\left(\nu + x^2\right)^{(\nu+1)/2}} \sum_{i=0}^{\infty} \Gamma\left((\nu + i + 1)/2\right)\left(\frac{\delta^i}{i!}\right)\left(\frac{2x^2}{\nu + x^2}\right)^{i/2} dx$$

where $\Gamma(\cdot)$ is the gamma function. The value of the distribution function at the point t_0 is the probability that the random variable takes a value less than or equal to t_0.

The noncentral *t* random variable can be defined by the distribution function above, or alternatively and equivalently, as the ratio of a normal random variable and an independent chi-squared random variable. If w has a normal distribution with mean δ and variance equal to one, u has an independent chi-squared distribution with ν degrees of freedom, and

$$x = w \Big/ \sqrt{u / \nu}$$

then x has a noncentral t distribution with ν degrees of freedom and noncentrality parameter δ.

The distribution function of the noncentral t can also be expressed as a double integral involving a normal density function (see, for example, Owen, 1962, page 108). The function `TNDF` uses the method of Owen (1962, 1965), which uses repeated integration by parts on that alternate expression for the distribution function.

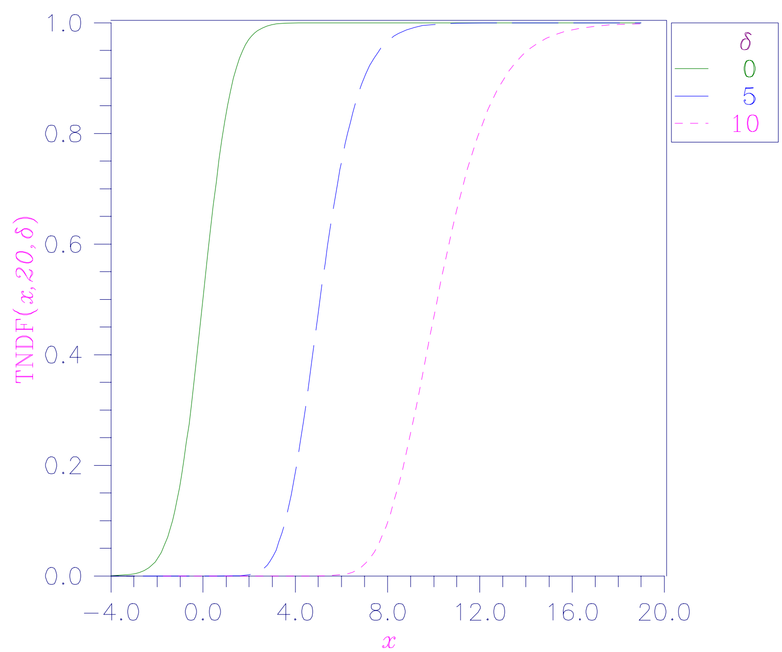

Figure 11-13   Noncentral Student's *t* Distribution Function

Example

Suppose T is a noncentral t random variable with 6 degrees of freedom and noncentrality parameter 6. In this example, we find the probability that T is less than 12.0. (This can be checked using the table on page 111 of Owen, 1962, with $\eta = 0.866$, which yields $\lambda = 1.664$.)

```
INTEGER    IDF, NOUT
REAL       DELTA, P, T, TNDF
```

```
      EXTERNAL   TNDF, UMACH
C
      CALL UMACH (2, NOUT)
      IDF   = 6
      DELTA = 6.0
      T     = 12.0
      P     = TNDF(T,IDF,DELTA)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that T is less than 12.0 is ', F6.4)
      END
```

Output

```
The probability that T is less than 12.0 is 0.9501
```

GCDF/DGCDF (Single/Double precision)

Evaluate a general continuous cumulative distribution function given ordinates of the density.

Usage

GCDF(X0, IOPT, M, X, F)

Arguments

X0 — Point at which the distribution function is to be evaluated.   (Input)

IOPT — Indicator of the method of interpolation.   (Input)

IOPT	**Interpolation Method**
1	Linear interpolation with equally spaced abscissas.
2	Linear interpolation with possibly unequally spaced abscissas.
3	A cubic spline is fitted to equally spaced abscissas.
4	A cubic spline is fitted to possibly unequally spaced abscissas.

M — Number of ordinates of the density supplied.   (Input)
M must be greater than 1 for linear interpolation (IOPT = 1 or 2) and greater than 3 if a curve is fitted through the ordinates (IOPT = 3 or 4).

X — Array containing the abscissas or the endpoints.   (Input)
If IOPT = 1 or 3, X is of length 2. If IOPT = 2 or 4, X is of length M. For IOPT = 1 or 3, X(1) contains the lower endpoint of the support of the distribution and X(2) is the upper endpoint. For IOPT = 2 or 4, X contains, in strictly increasing order, the abscissas such that X(I) corresponds to F(I).

F — Vector of length M containing the probability density ordinates corresponding to increasing abscissas.   (Input)
If IOPT = 1 or 3; for I = 1, 2, …, M, F(I) corresponds to X(1) + (I – 1) * (X(2) – X(1))/(M – 1); otherwise, F and X correspond one for one.

GCDF — Function value, the probability that a random variable whose density is given in F takes a value less than or equal to X0.   (Output)

Comments

If IOPT = 3, automatic workspace usage is

GCDF 6 * M units, or
DGCDF 11 * M units.

If IOPT = 4, automatic workspace usage is

GCDF 5 * M units, or
DGCDF 9 * M units.

Workspace may be explicitly provided, if desired, by the use of G4DF/DG4DF. The reference is

G4DF(P, IOPT, M, X, F, WK, IWK)

The arguments in addition to those of GCDF are

WK — Work vector of length 5 * M if IOPT = 3, and of length 4 * M if IOPT = 4.

IWK — Work vector of length M.

Algorithm

Function GCDF evaluates a continuous distribution function, given ordinates of the probability density function. It requires that the range of the distribution be specified in X. For distributions with infinite ranges, endpoints must be chosen so that most of the probability content is included. The function GCDF first fits a curve to the points given in X and F with either a piecewise linear interpolant or a C^1 cubic spline interpolant based on a method by Akima (1970). Function GCDF then determines the area, A, under the curve. (If the distribution were of finite range and if the fit were exact, this area would be 1.0.) Using the same fitted curve, GCDF next determines the area up to the point x_0 (= X0). The value returned is the area up to x_0 divided by A. Because of the scaling by A, it is not assumed that the integral of the density defined by X and F is 1.0.

For most distributions, it is likely that better approximations to the distribution function are obtained when IOPT equals 3 or 4, that is, when a cubic spline is used to approximate the function. It is also likely that better approximations can be obtained when the abscissas are chosen more densely over regions where the density and its derivatives (when they exist) are varying greatly.

Example

In this example, we evaluate the beta distribution function at the point 0.6. The probability density function of a beta random variable with parameters p and q is

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}(1-x)^{q-1} \quad \text{for } 0 \le x \le 1$$

where $\Gamma(\cdot)$ is the gamma function. The density is equal to 0 outside the interval [0, 1]. We compute a constant multiple (we can ignore the constant gamma functions) of the density at 300 equally spaced points and input this information in X and F. Knowing that the probability density of this distribution is very peaked in the vicinity of 0.5, we could perhaps get a better fit by using unequally spaced abscissas, but we will keep it simple. Note that this is the same example as one used in the description of routine BETDF (page 189). The result from BETDF would be expected to be more accurate than that from GCDF since BETDF is designed specifically for this distribution.

```
      INTEGER    M
      PARAMETER  (M=300)
C
      INTEGER    I, IOPT, NOUT
      REAL       F(M), GCDF, H, P, PIN1, QIN1, X(2), X0, XI
      EXTERNAL   GCDF, UMACH
C
      CALL UMACH (2, NOUT)
      X0   = 0.6
      IOPT = 3
C                                  Initializations for a beta(12,12)
C                                  distribution.
      PIN1 = 11.0
      QIN1 = 11.0
      XI   = 0.0
      H    = 1.0/(M-1.0)
      X(1) = XI
      F(1) = 0.0
      XI   = XI + H
C                                  Compute ordinates of the probability
C                                  density function.
      DO 10  I=2, M - 1
         F(I) = XI**PIN1*(1.0-XI)**QIN1
         XI   = XI + H
   10 CONTINUE
      X(2) = 1.0
      F(M) = 0.0
      P    = GCDF(X0,IOPT,M,X,F)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that X is less than 0.6 is ', F6.4)
      END
```

Output

```
The probability that X is less than 0.6 is 0.8364
```

GCIN/DGCIN (Single/Double precision)

Evaluate the inverse of a general continuous cumulative distribution function given ordinates of the density.

Usage

GCIN(P, IOPT, M, X, F)

Arguments

P — Probability for which the inverse of the distribution function is to be evaluated. (Input)
P must be in the open interval (0.0, 1.0).

IOPT — Indicator of the method of interpolation. (Input)

IOPT	Interpolation Method
1	Linear interpolation with equally spaced abscissas.
2	Linear interpolation with possibly unequally spaced abscissas.
3	A cubic spline is fitted to equally spaced abscissas.
4	A cubic spline is fitted to possibly unequally spaced abscissas.

M — Number of ordinates of the density supplied. (Input)
M must be greater than 1 for linear interpolation (IOPT = 1 or 2) and greater than 3 if a curve is fitted through the ordinates (IOPT = 3 or 4).

X — Array containing the abscissas or the endpoints. (Input)
If IOPT = 1 or 3, X is of length 2. If IOPT = 2 or 4, X is of length M. For IOPT = 1 or 3, X(1) contains the lower endpoint of the support of the distribution and X(2) is the upper endpoint. For IOPT = 2 or 4, X contains, in strictly increasing order, the abscissas such that X(I) corresponds to F(I).

F — Vector of length M containing the probability density ordinates corresponding to increasing abscissas. (Input)
If IOPT = 1 or 3, for I = 1, 2, …, M, F(I) corresponds to X(1) + (I – 1) * (X(2) – X(1))/(M – 1); otherwise, F and X correspond one for one.

GCIN — Function value. (Output)
The probability that a random variable whose density is given in F takes a value less than or equal to GCIN is P.

Comments

If IOPT = 3, automatic workspace usage is

GCIN 6 * M units, or
DGCIN 11 * M units.

If IOPT = 4, automatic workspace usage is

GCIN 5 * M units, or
DGCIN 9 * M units.

Workspace may be explicitly provided, if desired, by the use of G3IN/DG3IN. The reference is

```
G3IN(P, IOPT, M, X, F, WK, IWK)
```

The arguments in addition to those of GCIN are

WK — Work vector of length 5 * M if IOPT = 3, and of length 4 * M if IOPT = 4.

IWK — Work vector of length `M`.

Algorithm

Function `GCIN` evaluates the inverse of a continuous distribution function, given ordinates of the probability density function. The range of the distribution must be specified in `X`. For distributions with infinite ranges, endpoints must be chosen so that most of the probability content is included.

The function `GCIN` first fits a curve to the points given in `X` and `F` with either a piecewise linear interpolant or a C^1 cubic spline interpolant based on a method by Akima (1970). Function `GCIN` then determines the area, A, under the curve. (If the distribution were of finite range and if the fit were exact, this area would be 1.0.) It next finds the maximum abscissa up to which the area is less than AP and the minimum abscissa up to which the area is greater than AP. The routine then interpolates for the point corresponding to AP. Because of the scaling by A, it is not assumed that the integral of the density defined by `X` and `F` is 1.0.

For most distributions, it is likely that better approximations to the distribution function are obtained when `IOPT` equals 3 or 4, that is, when a cubic spline is used to approximate the function. It is also likely that better approximations can be obtained when the abscissas are chosen more densely over regions where the density and its derivatives (when they exist) are varying greatly.

Example

In this example, we find the 90-th percentage point for a beta random variable with parameters 12 and 12. The probability density function of a beta random variable with parameters p and q is

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}(1-x)^{q-1} \quad \text{for } 0 \le x \le 1$$

where $\Gamma(\cdot)$ is the gamma function. The density is equal to 0 outside the interval [0, 1]. With $p = q$, this is a symmetric distribution. Knowing that the probability density of this distribution is very peaked in the vicinity of 0.5, we could perhaps get a better fit by using unequally spaced abscissas, but we will keep it simple and use 300 equally spaced points. Note that this is the same example that is used in the description of routine `BETIN` (page 191). The result from `BETIN` would be expected to be more accurate than that from `GCIN` since `BETIN` is designed specifically for this distribution.

```
      INTEGER    M
      PARAMETER  (M=300)
C
      INTEGER    I, IOPT, NOUT
      REAL       BETA, C, F(M), GCIN, H, P, PIN, PIN1, QIN, QIN1,
     &           X(2), X0, XI
      EXTERNAL   BETA, GCIN, UMACH
C
      CALL UMACH (2, NOUT)
```

```
      P    = 0.9
      IOPT = 3
C                                  Initializations for a beta(12,12)
C                                  distribution.
      PIN  = 12.0
      QIN  = 12.0
      PIN1 = PIN - 1.0
      QIN1 = QIN - 1.0
      C    = 1.0/BETA(PIN,QIN)
      XI   = 0.0
      H    = 1.0/(M-1.0)
      X(1) = XI
      F(1) = 0.0
      XI   = XI + H
C                                  Compute ordinates of the probability
C                                  density function.
      DO 10  I=2, M - 1
         F(I) = C*XI**PIN1*(1.0-XI)**QIN1
         XI   = XI + H
   10 CONTINUE
      X(2) = 1.0
      F(M) = 0.0
      X0   = GCIN(P,IOPT,M,X,F)
      WRITE (NOUT,99999) X0
99999 FORMAT (' X is less than ', F6.4, ' with probability 0.9.')
      END
```

Output

```
X is less than 0.6304 with probability 0.9.
```

Chapter 12: Mathieu Functions

Routines

Evaluate the eigenvalues
for the periodic Mathieu functions .... MATEE 217
Evaluate even, periodic Mathieu functions.......................... MATCE 220
Evaluate odd, periodic Mathieu functions .... MATSE 223

Usage Notes

Mathieu's equation is

$$\frac{d^2y}{dv^2} + (a - 2q\cos 2v)y = 0$$

It arises from the solution, by separation of variables, of Laplace's equation in elliptical coordinates, where a is the separation constant and q is related to the ellipticity of the coordinate system. If we let $t = \cos v$, then Mathieu's equation can be written as

$$\left(1 - t^2\right)\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \left(a + 2q - 4qt^2\right)y = 0$$

For various physically important problems, the solution $y(t)$ must be periodic. There exist, for particular values of a, periodic solutions to Mathieu's equation of period $k\pi$ for any integer k. These particular values of a are called *eigenvalues* or *characteristic values*. They are computed using the routine `MATEE` (page 217).

There exist sequences of both even and odd periodic solutions to Mathieu's equation. The even solutions are computed by `MATCE` (page 220). The odd solutions are computed by `MATSE` (page 223).

MATEE/DMATEE (Single/Double precision)

Evaluate the eigenvalues for the periodic Mathieu functions.

Usage

`CALL MATEE (Q, N, ISYM, IPER, EVAL)`

Arguments

Q — Parameter. (Input)

N — Number of eigenvalues to be computed. (Input)

ISYM — Symmetry indicator. (Input)

ISYM	**Meaning**
0	Even
1	Odd

IPER — Periodicity indicator. (Input)

ISYM	**Period**
0	pi
1	2 * pi

EVAL — Vector of length `N` containing the eigenvalues. (Output)

Comments

1. Automatic workspace usage is

`MATEE`	2 * `N` units, or
`DMATEE`	4 * `N` units.

   Workspace may be explicitly provided, if desired, by use of `M2TEE`/`DM2TEE`. The reference is

   ```
   CALL M2TEE (Q, N, ISYM, IPER, EVAL, NORDER, WORKD,
               WORKE)
   ```

   The additional arguments are as follows:

   NORDER — Order of the matrix whose eigenvalues are computed. (Input)

   WORKD — Work vector of size `NORDER`. (Input/Output)
   If `EVAL` is large enough then `EVAL` and `WORKD` can be the same vector.

   WORKE — Work vector of size `NORDER`. (Input/Output)

2. Informational error

Type	Code	
4	1	The iteration for the eigenvalues did not converge.

Algorithm

The eigenvalues of Mathieu’s equation are computed by a method due to Hodge (1972). The desired eigenvalues are the same as the eigenvalues of the following symmetric, tridiagonal matrix:

$$\begin{bmatrix} W_0 & qX_0 & 0 & 0 & \dots \\ qX_0 & W_2 & qX_2 & 0 & \dots \\ 0 & qX_2 & W_4 & qX_4 & \dots \\ 0 & 0 & qX_4 & W_6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Here,

$$X_m = \begin{cases} \sqrt{2} & \text{if ISYM} = \text{IPER} = m = 0 \\ 1 & \text{otherwise} \end{cases}$$

$$W_m = \left[m + \text{IPER} + 2(1 - \text{IPER})\text{ISYM}\right]^2 + V_m$$

where

$$V_m = \begin{cases} +q & \text{if IPER} = 1, \text{ISYM} = 0 \text{ and } m = 0 \\ -q & \text{if IPER} = 1, \text{ISYM} = 1 \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since the above matrix is semi-infinite, it must be truncated before its eigenvalues can be computed. Routine `MATEE` computes an estimate of the number of terms needed to get accurate results. This estimate can be overridden by calling `M2TEE` with `NORDER` equal to the desired order of the truncated matrix.

The eigenvalues of this matrix are computed using the routine `EVLSB` found in the IMSL MATH/LIBRARY Chapter 2.

Example

In this example, the eigenvalues for $q = 5$, even symmetry, and π periodicity are computed and printed.

```
C                                 Declare variables
      INTEGER    N
      PARAMETER  (N=10)
C
      INTEGER    ISYM, IPER, K, NOUT
      REAL       Q, EVAL(N)
      EXTERNAL   CONST, MATEE, UMACH
C                                 Compute
      Q    = 5.0
      ISYM = 0
      IPER = 0
      CALL MATEE (Q, N, ISYM, IPER, EVAL)
C                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) 2*K-2, EVAL(K)
   10 CONTINUE
99999 FORMAT (' Eigenvalue', I2, ' = ', F9.4)
      END
```

Output

```
Eigenvalue 0 =   -5.8000
Eigenvalue 2 =    7.4491
Eigenvalue 4 =   17.0966
Eigenvalue 6 =   36.3609
Eigenvalue 8 =   64.1989
Eigenvalue10 =  100.1264
Eigenvalue12 =  144.0874
Eigenvalue14 =  196.0641
Eigenvalue16 =  256.0491
Eigenvalue18 =  324.0386
```

MATCE/DMATCE (Single/Double precision)

Evaluate a sequence of even, periodic, integer order, real Mathieu functions.

Usage

CALL MATCE (X, Q, N, CE)

Arguments

X — Argument for which the sequence of Mathieu functions is to be evaluated. (Input)

Q — Parameter. (Input)
The parameter Q must be positive.

N — Number of elements in the sequence. (Input)

CE — Vector of length N containing the values of the function through the series. (Output)
CE(I) contains the value of the Mathieu function of order I – 1 at X for I = 1 to N.

Comments

1. Automatic workspace usage is

   MATCE 6 * NORDER + 6 units, or
   DMATCE 12 * NORDER + 12 units.

   Workspace may be explicitly provided, if desired, by use of M2TCE/DM2TCE. The reference is

   ```
   CALL M2TCE (X, Q, N, CE, NORDER, NEEDEV, EVAL0,
               EVAL1, COEF, WORK, BSJ)
   ```

   The additional arguments are as follows:

   NORDER — Order of the matrix used to compute the eigenvalues. (Input)
   It must be greater than N. Routine MATSE computes NORDER by the following call to M3TEE.

```
CALL M3TEE(Q, N, NORDER)
```

NEEDEV — Logical variable, if `.TRUE.`, the eigenvalues must be computed. (Input)

EVAL0 — Real work vector of length `NORDER` containing the eigenvalues computed by `MATEE` with `ISYM` = 0 and `IPER` = 0. (Input/Output)
If `NEEDEV` is `.TRUE.`, then `EVAL0` is computed by `M2TCE`; otherwise, it must be set as an input value.

EVAL1 — Real work vector of length `NORDER` containing the eigenvalues computed by `MATEE` with `ISYM` = 0 and `IPER` = 1. (Input/Output)
If `NEEDEV` is `.TRUE.`, then `EVAL1` is computed by `M2TCE`; otherwise, it must be set as an input value.

COEF — Real work vector of length `NORDER` + 4.

WORK — Real work vector of length `NORDER` + 4.

BSJ — Real work vector of length 2 * `NORDER` – 2.

2. Informational error

Type	Code	
4	1	The iteration for the eigenvalues did not converge.

Algorithm

The eigenvalues of Mathieu's equation are computed using `MATEE` (page 217). The function values are then computed using a sum of Bessel functions, see Gradshteyn and Ryzhik (1965), equation 8.661.

Example 1

In this example, $ce_n(x = \pi/4, q = 1)$, $n = 0, \ldots, 9$ is computed and printed.

```
C                                  Declare variables
      INTEGER    N
      PARAMETER  (N=10)
C
      INTEGER    K, NOUT
      REAL       CE(N), CONST, Q, X
      EXTERNAL   CONST, MATCE, UMACH
C                                  Compute
      Q = 1.0
      X = 0.25*CONST('PI')
      CALL MATCE (X, Q, N, CE)
C                                  Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, X, Q, CE(K)
   10 CONTINUE
99999 FORMAT (' ce sub', I2, ' (', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
ce sub 0 ( 0.785, 1.000) =  0.654
ce sub 1 ( 0.785, 1.000) =  0.794
ce sub 2 ( 0.785, 1.000) =  0.299
ce sub 3 ( 0.785, 1.000) = -0.555
ce sub 4 ( 0.785, 1.000) = -0.989
ce sub 5 ( 0.785, 1.000) = -0.776
ce sub 6 ( 0.785, 1.000) = -0.086
ce sub 7 ( 0.785, 1.000) =  0.654
ce sub 8 ( 0.785, 1.000) =  0.998
ce sub 9 ( 0.785, 1.000) =  0.746
```

Example 2

In this example, we compute $\mathrm{ce}_n(x, q)$ for various values of n and x and a fixed value of q. To avoid having to recompute the eigenvalues, which depend on q but not on x, we compute the eigenvalues once and pass in their value to `M2TCE`. The eigenvalues are computed using `MATEE` (page 217). The routine `M3TEE` is used to compute `NORDER` based on `Q` and `N`. The arrays `BSJ`, `COEF` and `WORK` are used as temporary storage in `M2TCE`.

```
C                                 Declare variables
      INTEGER    MAXORD, N, NX
      PARAMETER  (MAXORD=100, N=4, NX=5)
C
      INTEGER    ISYM, K, NORDER, NOUT
      REAL       BSJ(2*MAXORD-2), CE(N), CONST, COEF(MAXORD+4)
      REAL       EVAL0(MAXORD), EVAL1(MAXORD), PI, Q, WORK(MAXORD+4), X
      EXTERNAL   CONST, MATEE, M2TCE, UMACH
C                                 Compute NORDER
      Q = 1.0
      CALL M3TEE (Q, N, NORDER)
C
      CALL UMACH (2, NOUT)
      WRITE (NOUT, 99997) NORDER
C                                 Compute eigenvalues
      ISYM = 0
      CALL MATEE (Q, NORDER, ISYM, 0, EVAL0)
      CALL MATEE (Q, NORDER, ISYM, 1, EVAL1)
C
      PI = CONST('PI')
C                                 Compute function values
      WRITE (NOUT, 99998)
      DO 10  K=0, NX
         X = (K*PI)/NX
         CALL M2TCE(X, Q, N, CE, NORDER, .FALSE., EVAL0, EVAL1,
     &     COEF, WORK, BSJ)
         WRITE (NOUT,99999) X, CE(1), CE(2), CE(3), CE(4)
   10 CONTINUE
C
99997 FORMAT (' NORDER = ', I3)
99998 FORMAT (/, 28X, 'Order', /, 20X, '0', 7X, '1', 7X,
     &     '2', 7X, '3')
99999 FORMAT (' ce(', F6.3, ') = ', 4F8.3)
      END
```

Output

```
NORDER =  23
                                   Order
                    0          1          2          3
ce( 0.000) =    0.385      0.856      1.086      1.067
ce( 0.628) =    0.564      0.838      0.574     -0.131
ce( 1.257) =    0.926      0.425     -0.575     -0.820
ce( 1.885) =    0.926     -0.425     -0.575      0.820
ce( 2.513) =    0.564     -0.838      0.574      0.131
ce( 3.142) =    0.385     -0.856      1.086     -1.067
```

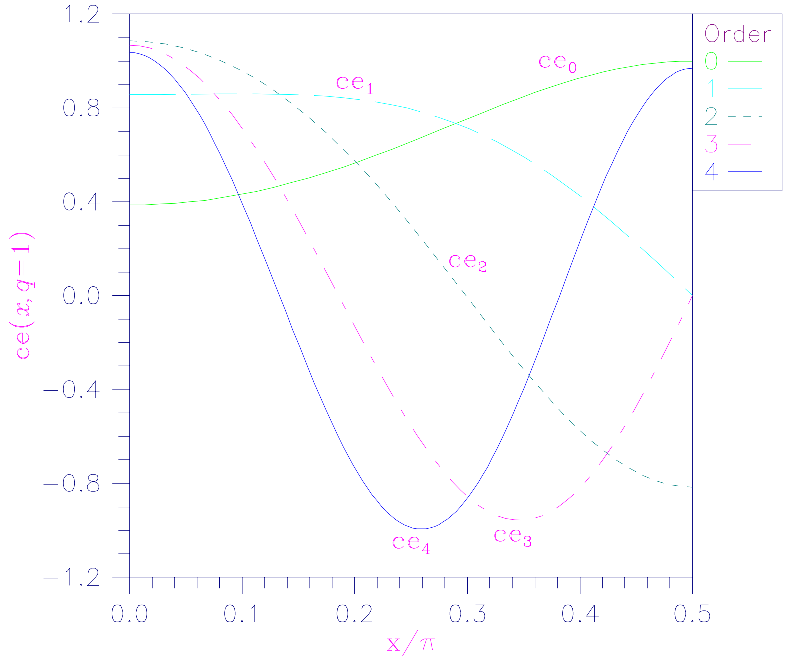

Figure 12-1   Plot of $ce_n(x, q = 1)$

MATSE/DMATSE (Single/Double precision)

Evaluate a sequence of odd, periodic, integer order, real Mathieu functions.

Usage

```
CALL MATSE (X, Q, N, SE)
```

Arguments

X — Argument for which the sequence of Mathieu functions is to be evaluated. (Input)

Q — Parameter. (Input)
The parameter `Q` must be positive.

N — Number of elements in the sequence. (Input)

SE — Vector of length `N` containing the values of the function through the series. (Output)
`SE(I)` contains the value of the Mathieu function of order `I` at `X` for `I` = 1 to `N`.

Comments

1. Automatic workspace usage is

   `MATSE` 6 * `NORDER` + 9 units, or
   `DMATSE` 12 * `NORDER` + 18 units.

   Workspace may be explicitly provided, if desired, by use of `M2TSE`/`DM2TSE`. The reference is

   ```
   CALL M2TSE (X, Q, N, SE, NORDER, NEEDEV, EVAL0,
               EVAL1, COEF, WORK, BSJ)
   ```

   The additional arguments are as follows:

   NORDER — Order of the matrix used to compute the eigenvalues. (Input)
   It must be greater than `N`. Routine `MATSE` computes `NORDER` by the following call to `M3TEE`.

   ```
   CALL M3TEE (Q, N, NORDER)
   ```

   NEEDEV — Logical variable, if `.TRUE.`, the eigenvalues must be computed. (Input)

   EVAL0 — Real work vector of length `NORDER` containing the eigenvalues computed by `MATEE` with `ISYM` = 1 and `IPER` = 0. (Input/Output)
   If `NEEDEV` is `.TRUE.`, then `EVAL0` is computed by `M2TSE`; otherwise, it must be set as an input value.

   EVAL1 — Real work vector of length `NORDER` containing the eigenvalues computed by `MATEE` with `ISYM` = 1 and `IPER` = 1. (Input/Output)
   If `NEEDEV` is `.TRUE.`, then `EVAL1` is computed by `M2TSE`; otherwise, it must be set as an input value.

   COEF — Real work vector of length `NORDER` + 4.

   WORK — Real work vector of length `NORDER` + 4.

   BSI — Real work vector of length 2 * `NORDER` + 1.

2. Informational error

Type	Code	
4	1	The iteration for the eigenvalues did not converge.

Algorithm

The eigenvalues of Mathieu's equation are computed using `MATEE` (page 217). The function values are then computed using a sum of Bessel functions, see Gradshteyn and Ryzhik (1965), equation 8.661.

Example

In this example, $\mathrm{se}_n(x = \pi/4, q = 10)$, $n = 0, \ldots, 9$ is computed and printed.

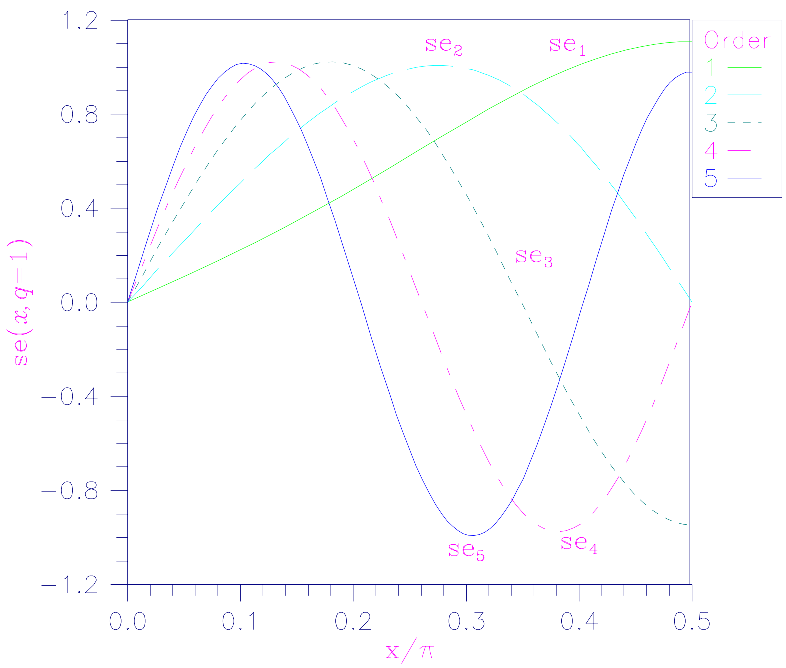

Figure 12-2 Plot of $\mathrm{se}_n(x, q = 1)$

```
C                                  Declare variables
      INTEGER    N
      PARAMETER  (N=10)
C
      INTEGER    K, NOUT
      REAL       SE(N), CONST, Q, X
      EXTERNAL   CONST, MATSE, UMACH
C                                  Compute
      Q = 10.0
      X = 0.25*CONST('PI')
      CALL MATSE (X, Q, N, SE)
C                                  Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, X, Q, SE(K)
   10 CONTINUE
99999 FORMAT (' se sub', I2, ' (', F6.3, ',', F6.3, ') = ', F6.3)
```

```
      END
```

Output

```
se sub 0 ( 0.785,10.000) =  0.250
se sub 1 ( 0.785,10.000) =  0.692
se sub 2 ( 0.785,10.000) =  1.082
se sub 3 ( 0.785,10.000) =  0.960
se sub 4 ( 0.785,10.000) =  0.230
se sub 5 ( 0.785,10.000) = -0.634
se sub 6 ( 0.785,10.000) = -0.981
se sub 7 ( 0.785,10.000) = -0.588
se sub 8 ( 0.785,10.000) =  0.219
se sub 9 ( 0.785,10.000) =  0.871
```

Chapter 13: Miscellaneous Functions

Routines

Spence dilogarithm .... SPENC 229
Initialize a Chebyshev series.. INITS 230
Evaluate a Chebyshev series.. CSEVL 231

Usage Notes

Many functions of one variable can be numerically computed using a Chebyshev series,

$$f(x) \ddagger \sum_{n=0}^{\infty} A_n T_n(x) \quad -1 \le x \le 1$$

A Chebyshev series is better for numerical computation than a Taylor series since the Chebyshev polynomials, $T_n(x)$, are better behaved than the monomials, x^n.

A Taylor series can be converted into a Chebyshev series using an algorithm of Fields and Wimp, (see Luke (1969), page 292).

Let

$$f(x) = \sum_{n=0}^{\infty} \xi_n x^n$$

be a Taylor series expansion valid for $|x| < 1$. Define

$$A_n = \frac{2}{4^n} \sum_{k=0}^{\infty} \frac{\left(n+\frac{1}{2}\right)_k (n+1)_k \xi_{n+k}}{(2n+1)_k k!}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is Pochhammer's symbol.

(Note that $(a)_{k+1} = (a+k)(a)_k$). Then,

$$f(x) = \tfrac{1}{2} T_0^*(x) + \sum_{n=1}^{\infty} A_n T_n^*(x) \quad 0 \le x \le 1$$

where

$$T_n^*(x)$$

are the shifted Chebyshev polynomials,

$$T_n^*(x) = T_n^*(2x-1)$$

In an actual implementation of this algorithm, the number of terms in the Taylor series and the number of terms in the Chebyshev series must both be finite. If the Taylor series is an alternating series, then the error in using only the first M terms is less than $|\xi_{M+1}|$. The error in truncating the Chebyshev series to N terms is no more than

$$\sum_{n=N+1}^{\infty} |c_n|$$

If the Taylor series is valid on $|x| < R$, then we can write

$$f(x) = \sum_{n=0}^{\infty} \xi_n R^n (x/R)^n$$

and use $\xi_n R^n$ instead of ξ_n in the algorithm to obtain a Chebyshev series in x/R valid for $0 < x < R$. Unfortunately, if R is large, then the Chebyshev series converges more slowly.

The Taylor series centered at zero can be shifted to a Taylor series centered at c. Let $t = x - c$, so

$$f(x) = f(t+c) = \sum_{n=0}^{\infty} \xi_n (t+c)^n = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \xi_n \binom{n}{j} c^{n-j} t^j$$

$$= \sum_{n=0}^{\infty} \hat{\xi}_n t^n = \sum_{n=0}^{\infty} \hat{\xi}_n (x-c)^n$$

By interchanging the order of the double sum, it can easily be shown that

$$\hat{\xi}_j = \sum_{n=j}^{\infty} \binom{n}{j} c^{n-j} \xi_n$$

By combining scaling and shifting, we can obtain a Chebyshev series valid over any interval $[a, b]$ for which the original Taylor series converges.

The algorithm can also be applied to asymptotic series,

$$f(x) \sim \sum_{n=0}^{\infty} \xi_n x^{-n} \text{ as } |x| \to \infty$$

by treating the series truncated to M terms as a polynomial in $1/x$. The asymptotic series is usually divergent; but if it is alternating, the error in truncating the series to M terms is less than $|\xi_{M+1}|/R^{M+1}$ for $R \le x < \infty$. Normally, as M increases, the error initially decreases to a small value and then increases without a bound. Therefore, there is a limit to the accuracy that can be obtained by increasing M. More accuracy can be obtained by increasing R. The optimal value of M depends on both the sequence ξ_j and R. For R fixed, the optimal value of M can be found by finding the value of M at which $|\xi_M|/R^M$ starts to increase.

Since we want a routine accurate to near machine precision, the algorithm must be implemented using somewhat higher precision than is normally used. This is best done using a symbolic computation package.

SPENC/DSPENC (Single/Double precision)

Evaluate a form of Spence's integral.

Usage

```
SPENC(X)
```

Arguments

X — Argument for which the function value is desired. (Input)

SPENC — Function value. (Output)

Algorithm

The Spence dilogarithm function, $s(x)$, is defined to be

$$s(x) = -\int_0^x \frac{\ln|1-y|}{y}\, dy$$

For $|x| \le 1$, the uniformly convergent expansion

$$s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

is valid.

Spence's function can be used to evaluate much more general integral forms. For example,

$$c\int_0^z \frac{\log(ax+b)}{cx+d}\, dx = \log\left|\frac{a(cz+d)}{ad-bc}\right| - s\left(\frac{a(cz+d)}{ad-bc}\right)$$

Example

In this example, $s(0.2)$ is computed and printed.

```
C                                 Declare variables
      INTEGER    NOUT
      REAL       SPENC, VALUE, X
      EXTERNAL   SPENC, UMACH
C                                 Compute
      X     = 0.2
      VALUE = SPENC(X)
C                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' SPENC(', F6.3, ') = ', F6.3)
      END
```

Output

```
SPENC( 0.200) = 0.211
```

INITS/INITDS (Single/Double precision)

Initialize the orthogonal series so the function value is the number of terms needed to insure the error is no larger than the requested accuracy.

Usage

`INITS(OS, NOS, ETA)`

Arguments

OS — Vector of length `NOS` containing coefficients in an orthogonal series. (Input)

NOS — Number of coefficients in `OS`. (Input)

ETA — Requested accuracy of the series. (Input)
Contrary to the usual convention, `ETA` is a `REAL` argument to `INITDS`.

INITS — Number of terms needed to insure the error is no larger than `ETA`. (Output)

Comments

`ETA` will usually be chosen to be one tenth of machine precision.

Algorithm

Function `INITS` initializes a Chebyshev series. The function `INITS` returns the number of terms in the series s of length n needed to insure that the error of the evaluated series is everywhere less than `ETA`. The number of input terms n must be greater than 1, so that a series of at least one term and an error estimate can be obtained. In addition, `ETA` should be larger than the absolute value of the last coefficient. If it is not, then all the terms of the series must be used, and no error estimate is available.

CSEVL/DCSEVL (Single/Double precision)

Evaluate the *N*-term Chebyshev series.

Usage

`CSEVL(X, CS, N)`

Arguments

X — Argument at which the series is to be evaluated. (Input)

CS — Vector of length `N` containing the terms of a Chebyshev series. (Input)
In evaluating `CS`, only half of the first coefficient is summed.

N — Number of terms in the vector `CS`. (Input)

CSEVL — Function value. (Output)

Comments

Informational error

Type	Code	
3	7	`X` is outside the interval (−1.1, +1.1)

Algorithm

Function `CSEVL` evaluates a Chebyshev series whose coefficients are stored in the array *s* of length *n* at the point *x*. The argument *x* must lie in the interval [−1, +1]. Other finite intervals can be linearly transformed to this canonical interval. Also, the number of terms in the series must be greater than zero but less than 1000. This latter limit is purely arbitrary; it is imposed in order to guard against the possibility of a floating point number being passed as an argument for *n*.

Reference Material

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User Errors

IMSL routines attempt to detect user errors and handle them in a way that provides as much information to the user as possible. To do this, we recognize various levels of severity of errors, and we also consider the extent of the error in the context of the purpose of the routine; a trivial error in one situation may be serious in another. IMSL routines attempt to report as many errors as they can reasonably detect. Multiple errors present a difficult problem in error detection because input is interpreted in an uncertain context after the first error is detected.

What Determines Error Severity

In some cases, the user's input may be mathematically correct, but because of limitations of the computer arithmetic and of the algorithm used, it is not possible to compute an answer accurately. In this case, the assessed degree of accuracy determines the severity of the error. In cases where the routine computes several output quantities, if some are not computable but most are, an error condition exists. The severity depends on an assessment of the overall impact of the error.

Terminal errors

If the user's input is regarded as meaningless, such as `N` = −1 when "`N`" is the number of equations, the routine prints a message giving the value of the erroneous input argument(s) and the reason for the erroneous input. The routine will then cause the user's program to stop. An error in which the user's input is meaningless is the most severe error and is called a *terminal error*. Multiple terminal error messages may be printed from a single routine.

Informational errors

In many cases, the best way to respond to an error condition is simply to correct the input and rerun the program. In other cases, the user may want to take actions in the program itself based on errors that occur. An error that may be used as the basis for corrective action within the program is called an *informational error*. If an informational error occurs, a user-retrievable code is set. A routine can return at most one informational error for a single reference to the routine. The codes for the informational error codes are printed in the error messages.

Other errors

In addition to informational errors, IMSL routines issue error messages for which no user-retrievable code is set. Multiple error messages for this kind of error may be printed. These errors, which generally are not described in the documentation, include terminal errors as well as less serious errors. Corrective action within the calling program is not possible for these errors.

Kinds of Errors and Default Actions

Five levels of severity of errors are defined in the MATH/LIBRARY Special Functions. Each level has an associated PRINT attribute and a STOP attribute. These attributes have default settings (YES or NO), but they may also be set by the user. The purpose of having multiple error severity levels is to provide independent control of actions to be taken for errors of different severity. Upon return from an IMSL routine, exactly one error state exists. (A code 0 “error” is no informational error.) Even if more than one informational error occurs, only one message is printed (if the PRINT attribute is YES). Multiple errors for which no corrective action within the calling program is reasonable or necessary result in the printing of multiple messages (if the PRINT attribute for their severity level is YES). Errors of any of the severity levels except level 5 may be informational errors.

Level 1: Note. A *note* is issued to indicate the possibility of a trivial error or simply to provide information about the computations. Default attributes: PRINT=NO, STOP=NO

Level 2: Alert. An *alert* indicates that the user should be advised about events occurring in the software. Default attributes: PRINT=NO, STOP=NO

Level 3: Warning. A *warning* indicates the existence of a condition that may require corrective action by the user or calling routine. A warning error may be issued because the results are accurate to only a few decimal places, because some of the output may be erroneous but most of the output is correct, or because some assumptions underlying the analysis technique are violated. Often no corrective action is necessary and the condition can be ignored. Default attributes: PRINT=YES, STOP=NO

Level 4: Fatal. A *fatal* error indicates the existence of a condition that may be serious. In most cases, the user or calling routine must take corrective action to recover. Default attributes: PRINT=YES, STOP=YES

Level 5: Terminal. A *terminal* error is serious. It usually is the result of an incorrect specification, such as specifying a negative number as the number of equations. These errors may also be caused by various programming errors impossible to diagnose correctly in FORTRAN. The resulting error message may be perplexing to the user. In such cases, the user is advised to compare carefully the actual arguments passed to the routine with the dummy argument descriptions given in the documentation. Special attention should be given to checking argument order and data types.

A terminal error is not an informational error because corrective action within the program is generally not reasonable. In normal usage, execution is terminated immediately when a terminal error occurs. Messages relating to more than one terminal error are printed if they occur. Default attributes: PRINT=YES, STOP=YES

The user can set PRINT and STOP attributes by calling `ERSET` as described in "Routines for Error Handling."

Errors in Lower-Level Routines

It is possible that a user's program may call an IMSL routine that in turn calls a nested sequence of lower-level IMSL routines. If an error occurs at a lower level in such a nest of routines and if the lower-level routine cannot pass the information up to the original user-called routine, then a traceback of the routines is produced. The only common situation in which this can occur is when an IMSL routine calls a user-supplied routine that in turn calls another IMSL routine.

Routines for Error Handling

There are three ways in which the user may interact with the IMSL error handling system: (1) to change the default actions, (2) to retrieve the integer code of an informational error so as to take corrective action, and (3) to determine the severity level of an error. The routines to use are `ERSET`, `IERCD`, and `N1RTY`, respectively.

ERSET

Change the default printing or stopping actions when errors of a particular error severity level occur.

Usage

```
CALL ERSET (IERSVR, IPACT, ISACT)
```

Arguments

IERSVR — Error severity level indicator. (Input)
If `IERSVR` = 0, actions are set for levels 1 to 5. If `IERSVR` is 1 to 5, actions are set for errors of the specified severity level.

IPACT — Printing action. (Input)

IPACT	**Action**
–1	Do not change current setting(s).
0	Do not print.
1	Print.
2	Restore the default setting(s).

ISACT — Stopping action. (Input)

ISACT	**Action**
–1	Do not change current setting(s).
0	Do not stop.
1	Stop.
2	Restore the default setting(s).

IERCD and N1RTY

The last two routines for interacting with the error handling system, `IERCD` and `N1RTY`, are `INTEGER` functions and are described in the following material.

`IERCD` retrieves the integer code for an informational error. Since it has no arguments, it may be used in the following way:

`ICODE` = `IERCD`()

The function retrieves the code set by the most recently called IMSL routine.

`N1RTY` retrieves the error type set by the most recently called IMSL routine. It is used in the following way:

`ITYPE` = `N1RTY`(1)

`ITYPE` = 1, 2, 4, and 5 correspond to error severity levels 1, 2, 4, and 5, respectively. `ITYPE` = 3 and `ITYPE` = 6 are both warning errors, error severity level 3. While `ITYPE` = 3 errors are informational errors (`IERCD`() ≠ 0), `ITYPE` = 6 errors are not informational errors (`IERCD`() = 0).

For software developers requiring additional interaction with the IMSL error handling system, see Aird and Howell (1991).

Examples

Changes to Default Actions

Some possible changes to the default actions are illustrated below. The default actions remain in effect for the kinds of errors not included in the call to `ERSET`.

To turn off printing of warning error messages:
```
CALL ERSET (3, 0, -1)
```

To stop if warning errors occur:
```
CALL ERSET (3, -1, 1)
```

To print all error messages:
```
CALL ERSET (0, 1, -1)
```

To restore all default settings:
```
CALL ERSET (0, 2, 2)
```

Automatic Workspace Allocation

FORTRAN subroutines that work with arrays as input and output often require extra arrays for use as workspace while doing computations or moving around data. IMSL routines generally do not require the user explicitly to allocate such arrays for use as workspace. On most systems the workspace allocation is handled transparently. The only limitation is the actual amount of memory available on the system.

On some systems the workspace is allocated out of a stack that is passed as a FORTRAN array in a named common block `WORKSP`. A very similar use of a workspace stack is described by Fox et al. (1978, pages 116−121). (For compatibility with older versions of the IMSL Libraries, space is allocated from the `COMMON` block, if possible.)

The arrays for workspace appear as arguments in lower-level routines. For example, the IMSL routine `BSJS` (page 103), which computes the values of first kind real order Bessel functions, needs arrays for workspace. `BSJS` allocates arrays from the common area and passes them to the lower-level routine `B2JS` that does the computations. This scheme for using lower-level routines is followed throughout the IMSL Libraries. The names of these routines have a “2” in the second position (or in the third position in double precision routines having a “`D`” prefix). The user can provide workspace explicitly and call directly the “2-level” routine, which is documented along with the main routine. In a very few cases, the 2-level routine allows additional options that the main routine does not allow.

Prior to returning to the calling program, a routine that allocates workspace generally deallocates that space so that it becomes available for use in other routines.

Changing the Amount of Space Allocated

This section is relevant only to those systems on which the transparent workspace allocator is not available.

By default, the total amount of space allocated in the common area for storage of numeric data is 5000 numeric storage units. (A numeric storage unit is the amount of space required to store an integer or a real number. By comparison, a double precision unit is twice this amount. Therefore, the total amount of space allocated in the common area for storage of numeric data is 2500 double precision units.) This space is allocated as needed for `INTEGER`, `REAL`, or other numeric data. For larger problems in which the default amount of workspace is insufficient, the user can change the allocation by supplying the FORTRAN statements to define the array in the named common block and by informing the IMSL workspace allocation system of the new size of the common array. To request 7000 units, the statements are

```
COMMON /WORKSP/ RWKSP
REAL RWKSP(7000)
CALL IWKIN(7000)
```

If an IMSL routine attempts to allocate workspace in excess of the amount available in the common stack, the routine issues a fatal error message that indicates how much space is needed and prints statements like those above to guide the user in allocating the necessary amount. The program below uses IMSL routine `BSJS` (page 103) to illustrate this feature.

This routine requires workspace that is just larger than twice the number of function values requested.

```
      INTEGER    N
      REAL       BS(10000), X, XNU
      EXTERNAL   BSJS
C                                  Set Parameters
      XNU = .5
      X   = 1.
      N   = 6000
      CALL BSJS (XNU, X, N, BS)
      END
```

Output

```
*** TERMINAL ERROR from BSJS.  Insufficient workspace for
***          current allocation(s). Correct by calling
***          IWKIN from main program with the three
***          following statements:  (REGARDLESS OF
***          PRECISION)
***                COMMON /WORKSP/  RWKSP
***                REAL RWKSP(12018)
***                CALL IWKIN(12018)

*** TERMINAL ERROR from BSJS.  The workspace requirement is
***          based on N =6000.
STOP
```

In most cases, the amount of workspace is dependent on the parameters of the problem so the amount needed is known exactly. In a few cases, however, the amount of workspace is dependent on the data (for example, if it is necessary to count all of the unique values in a vector). Thus, the IMSL routine cannot tell in advance exactly how much workspace is needed. In such cases, the error message printed is an estimate of the amount of space required.

Character Workspace

Since character arrays cannot be equivalenced with numeric arrays, a separate named common block `WKSPCH` is provided for character workspace. In most respects, this stack is managed in the same way as the numeric stack. The default size of the character workspace is 2000 character units. (A character unit is the amount of space required to store one character.) The routine analogous to `IWKIN` used to change the default allocation is `IWKCIN`.

Machine-Dependent Constants

The function subprograms in this section return machine-dependent information and can be used to enhance portability of programs between different computers. The routines `IMACH`, `AMACH` and `DMACH` describe the computer's arithmetic. The routine `UMACH` describes the input, output and error output unit numbers.

INTEGER FUNCTION IMACH(I)

`IMACH` retrieves machine integer constants which define the arithmetic used by the computer.

`IMACH`(1) = Number of bits per integer storage unit.
`IMACH`(2) = Number of characters per integer storage unit.

Integers are represented in M-digit, base A form as

$$\sigma \sum_{k=0}^{M} x_k A^k$$

where σ is the sign and $0 \le x_k < A$, $k = 0, \dots, M$. Then,

`IMACH`(3) = A, the base.
`IMACH`(4) = M, the number of base-A digits.
`IMACH`(5) = $A^M - 1$, the largest integer.

The machine model assumes that floating-point numbers are represented in N-digit, base B form as

$$\sigma B^E \sum_{k=1}^{N} x_k B^{-k}$$

where σ is the sign and $0 \le x_k < B$, $k = 1, \dots, N$ and $E_{\min} \le E \le E_{\max}$ Then,

`IMACH`(6) = B	the base
`IMACH`(7) = N_s	the number of base-B digits in single precision
$\mathrm{IMACH}(8) = E_{\min_s}$	the smallest single precision exponent
$\mathrm{IMACH}(9) = E_{\max_s}$	the largest single precision exponent
`IMACH`(10) = N_d	the number of base-B digits in double precision

$$\mathrm{IMACH}(11) = E_{\min_d}$$ the smallest double precision exponent

$$\mathrm{IMACH}(12) = E_{\max_d}$$ the number of base-B digits in double precision

REAL FUNCTION AMACH(I)

The function subprogram `AMACH` retrieves real machine constants that define the computer's real or single-precision arithmetic. Such floating-point numbers are represented in N_s-digit, base B form as

$$\sigma B^E \sum_{k=1}^{N_s} x_k B^{-k}$$

where σ is the sign, $0 \le x_k < B$, $k = 1, \ldots, N_s$ and

$$E_{\min_s} \le E \le E_{\max_s}$$

Note that B = `IMACH`(6), N_s = `IMACH`(7),

$$E_{\min_s} = \mathrm{IMACH}(8), \text{ and } E_{\max_s} = \mathrm{IMACH}(9)$$

The IEEE standard for binary arithmetic (see IEEE 1985) specifies *quiet* NaN (not a number) as the result of various invalid or ambiguous operations, such as 0/0. The intent is that `AMACH`(6) return a *signaling* NaN. On IEEE format computers that do not support signaling NaN, a quiet NaN is returned. If the machine does not support a NaN, a special value near `AMACH`(2) is returned for `AMACH`(6). On computers that do not have a special representation for infinity, `AMACH`(2) returns the same value as `AMACH`(7).

`AMACH` is defined by the following table:

$\mathrm{AMACH}(1) = B^{E_{\min_s} - 1}$	the smallest positive number
$\mathrm{AMACH}(2) = B^{E_{\max_s}}\left(1 - B^{-N_s}\right)$	the largest number
$\mathrm{AMACH}(3) = B^{-N_s}$	the smallest relative spacing
$\mathrm{AMACH}(4) = B^{1-N_s}$	the largest relative spacing
`AMACH`(5) = $\log_{10}(B)$	
`AMACH`(6) = NaN	(signaling not a number)
`AMACH`(7) = positive machine infinity	
`AMACH`(7) = negative machine infinity	

DOUBLE PRECISION FUNCTION DMACH(I)

The function subprogram `DMACH` retrieves real machine constants that define the computer's double precision arithmetic. Such double-precision floating-point numbers are represented in N_d-digit, base B form as

$$\sigma B^E \sum_{k=1}^{N_d} x_k B^{-k}$$

where σ is the sign, $0 \le x_k < B$, $k = 0, \ldots, N_d$ and

$$E_{\min_d} \le E \le E_{\max_d}$$

Note that B = `IMACH`(6), N_d = `IMACH`(10),

$$E_{\min_d} = \text{IMACH}(11), \text{ and } E_{\max_d} = \text{IMACH}(12)$$

The IEEE standard for binary arithmetic (see IEEE 1985) specifies quiet NaN (not a number) as the result of various invalid or ambiguous operations, such as 0/0. The intent is that `DMACH`(6) return a *signaling* NaN. On IEEE format computers that do not support signaling NaN, a quiet NaN is returned. If the machine does not support a NaN, a special value near `DMACH`(2) is returned for `DMACH`(6). On computers that do not have a special representation for infinity, `DMACH`(2) = `DMACH`(7).

`DMACH` is defined by the following table:

$\text{DMACH}(1) = B^{E_{\min_d} - 1}$	the smallest positive number
$\text{DMACH}(2) = B^{E_{\max_d}}\left(1 - B^{-N_d}\right)$	the largest number
$\text{DMACH}(3) = B^{-N_d}$	the smallest relative spacing
$\text{DMACH}(4) = B^{1-N_d}$	the largest relative spacing
`DMACH`(5) = $\log_{10}(B)$	
`DMACH`(6) = NaN	(signaling not a number)
`DMACH`(7) = positive machine infinity	
`DMACH`(7) = negative machine infinity	

LOGICAL FUNCTION IFNAN(X), DIFNAN(DX)

The logical function `IFNAN` checks if the `REAL` argument `X` is NaN (not a number). Similarly, `DIFNAN` checks if the `DOUBLE PRECISION` argument `DX` is NaN.

The functions `IFNAN` and `DIFNAN` are provided to facilitate the transfer of programs across computer systems. This is because the check for NaN can be tricky and not portable across computer systems that do not adhere to the IEEE standard. For example, on computers that support the IEEE standard for binary arithmetic (see IEEE 1985), NaN is specified as a bit format not equal to itself. Thus the check is performed as

```
IFNAN = X .NE. X
```

On other computers that do not use IEEE floating point format, the check can be performed in single precision as

```
IFNAN = X .EQ. AMACH(6)
```

The function IFNAN or DIFNAN is equivalent to the specification of the function Isnan listed in the Appendix, (IEEE 1985). The following example illustrates the use of IFNAN. If X is NaN, a message is printed instead of X. (IMSL routine UMACH is used to retrieve the output unit number for printing the message.)

```
      INTEGER    NOUT
      REAL       AMACH, X
      LOGICAL    IFNAN
      EXTERNAL   AMACH, IFNAN, UMACH
C
      CALL UMACH (2, NOUT)
C
      X = AMACH(6)
      IF (IFNAN(X)) THEN
      WRITE (NOUT,*) ' X is NaN (not a number).'
      ELSE
      WRITE (NOUT,*) ' X = ', X
      END IF
C
      END
```

Output

```
X is NaN (not a number).
```

SUBROUTINE UMACH(N, NUNIT)

Routine UMACH sets or retrieves the input or output device unit numbers. UMACH is set automatically so that the default FORTRAN unit numbers for standard input and output are used. These unit numbers can be changed by inserting a call to UMACH at the beginning of the main program that calls MATH/LIBRARY Special Functions routines. If the input or output numbers are changed from the standard values, the user should insert an appropriate OPEN statement in the calling program.

The calling sequence for UMACH is

```
CALL UMACH (N, NUNIT)
```

where NUNIT is the input or output unit number that is either retrieved or set, depending on which value of N is selected.

The arguments are summarized by the following table:

N	Effect
1	Retrieves input unit number in NUNIT.
2	Retrieves output unit number in NUNIT.
3	Retrieves error output unit number in NUNIT.
–1	Sets the input unit number to NUNIT.
–2	Sets the output unit number to NUNIT.
–3	Sets the error output unit number to NUNIT.

If the value of N is negative, the input or output unit number is reset to NUNIT. If the value of N is positive, the input or output unit number is returned in NUNIT. In the following example, a terminal error is issued from the MATH/LIBRARY Special Functions AMACH function since the argument is invalid. With a call to UMACH, the error message will be written to a local file named 'CHECKERR'.

```
      INTEGER     N, AMACH
      REAL        X
      EXTERNAL   AMACH, UMACH
C                                       Set Parameter
      N = 0
C
      CALL UMACH (-3, 9)
      OPEN (UNIT=9,FILE='CHECKERR')
      X = AMACH(N)
      END
```

The output from this example, written to 'CHECKERR' is:

```
*** TERMINAL ERROR 5 from AMACH. The argument must be
***           between 1 and 8 inclusive. N = 0
```

Reserved Names

When writing programs accessing IMSL MATH/LIBRARY Special Functions, the user should choose FORTRAN names that do not conflict with names of IMSL subroutines, functions, or named common blocks, such as the workspace common block WORKSP (see page 237). The user needs to be aware of two types of name conflicts that can arise. The first type of name conflict occurs when a name (technically a *symbolic name*) is not uniquely defined within a program unit (either a main program or a subprogram). For example, such a name conflict exists when the name BSJS is used to refer both to a type REAL variable and to the IMSL routine BSJS in a single program unit. Such errors are detected during compilation and are easy to correct. The second type of name conflict, which can be more serious, occurs when names of program units and named common blocks are not unique. For example, such a name conflict would be caused by the user defining a routine named WORKSP and also referencing a MATH/LIBRARY Special Functions routine that uses the named common block WORKSP. Likewise, the user must not define a subprogram with the same name as a subprogram in MATH/LIBRARY Special Functions, that is referenced directly by the user's program or is referenced indirectly by other MATH/LIBRARY Special Functions subprograms.

MATH/LIBRARY Special Functions consists of many routines, some that are described in the *User's Manual* and others that are not intended to be called by the user and, hence, that are not documented. If the choice of names were completely random over the set of valid FORTRAN names and if a program uses only a small subset of MATH/LIBRARY Special Functions, the probability of name conflicts is very small. Since names are usually chosen to be mnemonic, however, the user may wish to take some precautions in choosing FORTRAN names.

Many IMSL names consist of a root name that may have a prefix to indicate the type of the routine. For example, the IMSL single precision routine for computing Bessel functions of the first kind with real order has the name BSJS, which is the root name, and the corresponding IMSL double precision routine has the name DBSJS. Associated with these two routines are B2JS and DB2JS. BSJS and DBSJS are listed in the Alphabetical Index of Routines, but B2JS and DB2JS are not. The user of BSJS must consider both names BSJS and B2JS to be reserved; likewise, the user of DBSJS must consider both names DBSJS and DB2JS to be reserved. The names of *all* routines and named common blocks that are used by MATH/LIBRARY Special Functions and that do not have a numeral in the second position of the root name are listed in the Alphabetical Index of Routines. Some of the routines in this Index are not intended to be called by the user and so are not documented.

The careful user can avoid any conflicts with IMSL names if the following rules are observed:

- Do not choose a name that appears in the Alphabetical Index of Routines in the *User's Manual*.
- Do not choose a name of three or more characters with a numeral in the second or third position.
- Do not construct a name by replacing the leading "C" of a MATH/LIBRARY Special Functions routine name with a "Z." For example, users should not select the name "ZCOS" because CCOS is a MATH/LIBRARY Special Functions routine.

These simplified rules include many combinations that are, in fact, allowable. However, if the user selects names that conform to these rules, no conflict will be encountered.

Deprecated and Deleted Routines

The routines in the following list are being deprecated in Version 2.0 of MATH/LIBRARY Special Functions. A deprecated routine is one that is no longer used by anything in the library but is being included in the product for those users who may be currently referencing it in their application. However, any future versions of MATH/LIBRARY Special Functions will not include these routines. If any of these routines are being called within an application, it is recommended that you change your code or retain the deprecated routine before replacing this library with the next version. Most of these routines were called by users only when they needed to set up their own workspace. Thus, the impact of these changes should be limited.

```
G2DF
G2IN
G3DF
```

The following FORTRAN intrinsic functions are no longer supplied by IMSL. They can all be found in their manufacturer's FORTRAN runtime libraries. If any change must be made to the user's application as a result of their removal from the IMSL Libraries, it is limited to the redeclaration of the function from "external" to "intrinsic." Argument lists and results should be identical.

ACOS	CEXP	DATAN2	DSQRT
AINT	CLOG	DCOS	DTAN
ALOG	COS	DCOSH	DTANH
ALOG10	COSH	DEXP	EXP
ASIN	CSIN	DINT	SIN
ATAN	CSQRT	DLOG	SINH
ATAN2	DACOS	DLOG10	SQRT
CABS	DASIN	DSIN	TAN
CCOS	DATAN	DSINH	TANH

Appendix A: GAMS Index

Description

This index lists routines in MATH/LIBRARY Special Functions by a tree-structured classification scheme known as GAMS. Boisvert, Howe, Kahaner, and Springmann (1990) give the GAMS classification scheme. The classification scheme given here is Version 2.0. The first level of the classification scheme is denoted by a letter A thru Z as follows:

A. Arithmetic, Error Analysis
B. Number Theory
C. Elementary and Special Functions
D. Linear Algebra
E. Interpolation
F. Solution of Nonlinear Equations
G. Optimization
H. Differentiation and Integration
I. Differential and Integral Equations
J. Integral Transforms
K. Approximation
L. Statistics, Probability
M. Simulation, Stochastic Modeling
N. Data Handling
O. Symbolic Computation
P. Computational Geometry
Q. Graphics
R. Service Routines
S. Software Development Tools
Z. Other

There are seven levels in the classification scheme. Subclasses for levels 3, 5, and 7 are denoted by letters “a” thru “w”. Subclasses for levels 2, 4, and 6 are denoted by the numbers 1 thru 23.

The index given in the following pages lists routines in MATH/LIBRARY Special Functions within each GAMS subclass. The purpose of the routine appear alongside the routine name.

IMSL MATH/LIBRARY Special Functions

C .... ELEMENTARY AND SPECIAL FUNCTIONS (*search also class L5*)

C1 .... Integer-valued functions (e.g., floor, ceiling, factorial, binomial coefficient)

`BINOM` Evaluate the binomial coefficient.
`FAC` Evaluate the factorial of the argument.

C2 .... Powers, roots, reciprocals

`CBRT` Evaluate the cube root.
`CCBRT` Evaluate the complex cube root.

C3 .... Polynomials

C3a....... Orthogonal

`INITS` Initialize the orthogonal series so the function value is the number of terms needed to insure the error is no larger than the requested accuracy.

C3a2..... Chebyshev, Legendre

`CSEVL` Evaluate the *N*-term Chebyshev series.

C4 .... Elementary transcendental functions

C4a....... Trigonometric, inverse trigonometric

`CACOS` Evaluate the complex arc cosine.
`CARG` Evaluate the argument of a complex number.
`CASIN` Evaluate the complex arc sine.
`CATAN` Evaluate the complex arc tangent.
`CATAN2` Evaluate the complex arc tangent of a ratio.
`CCOT` Evaluate the complex cotangent.
`COSDG` Evaluate the cosine for the argument in degrees.
`COT` Evaluate the cotangent.
`SINDG` Evaluate the sine for the argument in degrees.

C4b ...... Exponential, logarithmic

`ALNREL` Evaluate the natural logarithm of one plus the argument.
`CEXPRL` Evaluate the complex exponential function factored from first order.
`CLNREL` Evaluate the principal value of the complex natural logarithm of one plus the argument.
`CLOG10` Evaluate the principal value of the complex common logarithm.
`EXPRL` Evaluate the exponential function factored from first order, (`EXP(X)` – 1.0)/`X`.

C4c....... Hyperbolic, inverse hyperbolic

`ACOSH` Evaluate the arc hyperbolic cosine.

`ASINH` Evaluate the arc hyperbolic sine.
`ATANH` Evaluate the arc hyperbolic tangent.
`CACOSH` Evaluate the complex arc hyperbolic cosine.
`CASINH` Evaluate the complex arc hyperbolic sine.
`CATANH` Evaluate the complex arc hyperbolic tangent.
`CCOSH` Evaluate the complex hyperbolic cosine.
`CSINH` Evaluate the complex hyperbolic sine.
`CTAN` Evaluate the complex tangent.
`CTANH` Evaluate the complex hyperbolic tangent.

C5 .... Exponential and logarithmic integrals

`ALI` Evaluate the logarithmic integral.
`CHI` Evaluate the hyperbolic cosine integral.
`CI` Evaluate the cosine integral.`CIN` Evaluate a function closely related to the cosine integral.
`CINH` Evaluate a function closely related to the hyperbolic cosine integral.
`E1` Evaluate the exponential integral for arguments greater than zero and the Cauchy principal value of the integral for arguments less than zero.
`EI` Evaluate the exponential integral for arguments greater than zero and the Cauchy principal value for arguments less than zero.
`ENE` Evaluate the exponential integral of integer order for arguments greater than zero scaled by `EXP(X)`.
`SHI` Evaluate the hyperbolic sine integral.
`SI` Evaluate the sine integral.

C7 .... Gamma

C7a ...... Gamma, log gamma, reciprocal gamma

`ALGAMS` Return the logarithm of the absolute value of the gamma function and the sign of gamma.
`ALNGAM` Evaluate the logarithm of the absolute value of the gamma function.
`CGAMMA` Evaluate the complex gamma function.
`CGAMR` Evaluate the reciprocal complex gamma function.
`CLNGAM` Evaluate the complex natural logarithm of the gamma function.
`GAMMA` Evaluate the complete gamma function.
`GAMR` Evaluate the reciprocal gamma function.
`POCH` Evaluate a generalization of Pochhammer's symbol.
`POCH1` Evaluate a generalization of Pochhammer's symbol starting from the first order.

C7b ...... Beta, log beta

`ALBETA` Evaluate the natural logarithm of the complete beta function for positive arguments.

BETA Evaluate the complete beta function.
CBETA Evaluate the complex complete beta function.
CLBETA Evaluate the complex logarithm of the complete beta function.

C7c....... Psi function

CPSI Evaluate the logarithmic derivative of the gamma function for a complex argument.
PSI Evaluate the logarithmic derivative of the gamma function.

C7e....... Incomplete gamma

CHIDF Evaluate the chi-squared distribution function.
CHIIN Evaluate the inverse of the chi-squared distribution function.
GAMDF Evaluate the gamma distribution function.
GAMI Evaluate the incomplete gamma function.
GAMIC Evaluate the complementary incomplete gamma function.
GAMIT Evaluate the Tricomi form of the incomplete gamma function.

C7f .... Incomplete beta

BETAI Evaluate the incomplete beta function ratio.
BETDF Evaluate the beta probability distribution function.
BETIN Evaluate the inverse of the beta distribution function.

C8 .... Error functions

C8a....... Error functions, their inverses, integrals, including the normal distribution function

ANORDF Evaluate the standard normal (Gaussian) distribution function.
ANORIN Evaluate the inverse of the standard normal (Gaussian) distribution function.
CERFE Evaluate the complex scaled complemented error function.
ERF Evaluate the error function.
ERFC Evaluate the complementary error function.
ERFCE Evaluate the exponentially scaled complementary error function.
ERFCI Evaluate the inverse complementary error function.
ERFI Evaluate the inverse error function.

C8b ...... Fresnel integrals

FRESC Evaluate the cosine Fresnel integral.
FRESS Evaluate the sine Fresnel integral.

C8c....... Dawson’s integral

DAWS Evaluate Dawson function.

C10 ...... Bessel functions

C10a .... J, Y, $H^{(1)}$; $H^{(2)}$

C10a1 .. Real argument, integer order

`BSJ0`	Evaluate the Bessel function of the first kind of order zero.
`BSJ1`	Evaluate the Bessel function of the first kind of order one.
`BSJNS`	Evaluate a sequence of Bessel functions of the first kind with integer order and real arguments.
`BSY0`	Evaluate the Bessel function of the second kind of order zero.
`BSY1`	Evaluate the Bessel function of the second kind of order one.

C10a2 .. Complex argument, integer order

`CBJNS`	Evaluate a sequence of Bessel functions of the first kind with integer order and complex arguments.

C10a3 .. Real argument, real order

`BSJS`	Evaluate a sequence of Bessel functions of the first kind with real order and real positive arguments.
`BSYS`	Evaluate a sequence of Bessel functions of the second kind with real nonnegative order and real positive arguments.

C10a4 .. Complex argument, real order

`CBJS`	Evaluate a sequence of Bessel functions of the first kind with real order and complex arguments.
`CBYS`	Evaluate a sequence of Bessel functions of the second kind with real order and complex arguments.

C10b .... I, K

C10b1 .. Real argument, integer order

`BSI0`	Evaluate the modified Bessel function of the first kind of order zero.
`BSI0E`	Evaluate the exponentially scaled modified Bessel function of the first kind of order zero.
`BSI1`	Evaluate the modified Bessel function of the first kind of order one.
`BSI1E`	Evaluate the exponentially scaled modified Bessel function of the first kind of order one.
`BSINS`	Evaluate a sequence of Modified Bessel functions of the first kind with integer order and real arguments.
`BSK0`	Evaluate the modified Bessel function of the third kind of order zero.
`BSK0E`	Evaluate the exponentially scaled modified Bessel function of the third kind of order zero.
`BSK1`	Evaluate the modified Bessel function of the third kind of order one.

BSK1E Evaluate the exponentially scaled modified Bessel function of the third kind of order one.

C10b2 .. Complex argument, integer order

CBINS Evaluate a sequence of Modified Bessel functions of the first kind with integer order and complex arguments.

C10b3 Real argument, real order

BSIES Evaluate a sequence of exponentially scaled Modified Bessel functions of the first kind with nonnegative real order and real positive arguments.

BSIS Evaluate a sequence of Modified Bessel functions of the first kind with real order and real positive arguments.

BSKES Evaluate a sequence of exponentially scaled modified Bessel functions of the third kind of fractional order.

BSKS Evaluate a sequence of modified Bessel functions of the third kind of fractional order.

C10b4 .. Complex argument, real order

CBIS Evaluate a sequence of Modified Bessel functions of the first kind with real order and complex arguments.

CBKS Evaluate a sequence of Modified Bessel functions of the second kind with real order and complex arguments.

C10c..... Kelvin functions

AKEI0 Evaluate the Kelvin function of the second kind, kei, of order zero.

AKEI1 Evaluate the Kelvin function of the second kind, kei, of order one.

AKEIP0 Evaluate the Kelvin function of the second kind, kei, of order zero.

AKER0 Evaluate the Kelvin function of the second kind, ker, of order zero.

AKER1 Evaluate the Kelvin function of the second kind, ker, of order one.

AKERP0 Evaluate the derivative of the Kelvin function of the second kind, ker, of order zero.

BEI0 Evaluate the Kelvin function of the first kind, bei, of order zero.

BEI1 Evaluate the Kelvin function of the first kind, bei, of order one.

BEIP0 Evaluate the derivative of the Kelvin function of the first kind, bei, of order zero.

BER0 Evaluate the Kelvin function of the first kind, ber, of order zero.

BER1 Evaluate the Kelvin function of the first kind, ber, of order one.

`BERP0`	Evaluate the derivative of the Kelvin function of the first kind, ber, of order zero.

C10d .... Airy and Scorer functions

AI	Evaluate the Airy function.
`AID`	Evaluate the derivative of the Airy function.
`AIDE`	Evaluate the exponentially scaled derivative of the Airy function.
`AIE`	Evaluate the exponentially scaled Airy function.
`BI`	Evaluate the Airy function of the second kind.
`BID`	Evaluate the derivative of the Airy function of the second kind.
`BIDE`	Evaluate the exponentially scaled derivative of the Airy function of the second kind.
`BIE`	Evaluate the exponentially scaled Airy function of the second kind.

C14 ...... Elliptic integrals

`CEJCN`	Evaluate the complex Jacobi elliptic integral cn(z, m).
`CEJDN`	Evaluate the complex Jacobi elliptic integral dn(z, m).
`CEJSN`	Evaluate the complex Jacobi elliptic function sn(z, m).
`EJCN`	Evaluate the Jacobi elliptic function cn(x, m).
`EJDN`	Evaluate the Jacobi elliptic function dn(x, m).
`EJSN`	Evaluate the Jacobi elliptic function sn(x, m).
`ELE`	Evaluate the complete elliptic integral of the second kind $E(x)$.
`ELK`	Evaluate the complete elliptic integral of the kind $K(x)$.
`ELRC`	Evaluate an elementary integral from which inverse circular functions, logarithms and inverse hyperbolic functions can be computed.
`ELRD`	Evaluate Carlson's incomplete elliptic integral of the second kind `RD`(`X`, `Y`, `Z`).
`ELRF`	Evaluate Carlson's incomplete elliptic integral of the first kind `RF`(`X`, `Y`, `Z`).
`ELRJ`	Evaluate Carlson's incomplete elliptic integral of the third kind `RJ`(`X`, `Y`, `Z`, `RHO`).

C15 ...... Weierstrass elliptic functions

`CWPL`	Evaluate the Weierstrass *P*-function in the lemniscat case for complex argument with unit period parallelogram.
`CWPLD`	Evaluate the first derivative of the Weierstrass *P*-function in the lemniscatic case for complex argum with unit period parallelogram.
`CWPQ`	Evaluate the Weierstrass *P*-function in the equianharmonic case for complex argument with unit period parallelogram.

CWPQD Evaluate the first derivative of the Weierstrass *P*-function in the equianharmonic case for complex argument with unit period parallelogram.

C17 ...... Mathieu functions

MATCE Evaluate a sequence of even, periodic, integer order, real Mathieu functions.

MATEE Evaluate the eigenvalues for the periodic Mathieu functions.

MATSE Evaluate a sequence of odd, periodic, integer order, real Mathieu functions.

C19 ...... Other special functions

SPENC Evaluate a form of Spence's integral.

L........... STATISTICS, PROBABILITY

L5......... Function evaluation (*search also class C*)

L5a....... Univariate

L5a1..... Cumulative distribution functions, probability density functions

GCDF Evaluate a general continuous cumulative distribution function given ordinates of the density.

L5a1b... Beta, binomial

BETDF Evaluate the beta probability distribution function.

BINDF Evaluate the binomial distribution function.

BINPR Evaluate the binomial probability function.

CHIDF Evaluate the chi-squared distribution function.

CSNDF Evaluate the noncentral chi-squared distribution function.

L5a1f.... F distribution

FDF Evaluate the *F* distribution function.

L5a1g... Gamma, general, geometric

GAMDF Evaluate the gamma distribution function.

L5a1h... Halfnormal, hypergeometric

HYPDF Evaluate the hypergeometric distribution function.

HYPPR Evaluate the hypergeometric probability function.

L5a1k... Kendall *F* statistic, Kolmogorov-Smirnov

AKS1DF Evaluate the distribution function of the one-sided Kolmogorov-Smirnov goodness of fit D^+ or D^- test statistic based on continuous data for one sample.

AKS2DF Evaluate the distribution function of the Kolmogorov-Smirnov goodness of fit *D* test statistic based on continuous data for two samples.

L5a1n... Negative binomial, normal

`ANORDF` Evaluate the standard normal (Gaussian) distribution function.

L5a1p... Pareto, Poisson

`POIDF` Evaluate the Poisson distribution function.

`POIPR` Evaluate the Poisson probability function.

L5a1t.... t distribution

`TDF` Evaluate the Student's *t* distribution function.

`TNDF` Evaluate the noncentral Student's *t* distribution function.

L5a2..... Inverse cumulative distribution functions, sparsity functions

`GCIN` Evaluate the inverse of a general continuous cumulative distribution function given ordinates of the density.

L5a2b... Beta, binomial

`BETIN` Evaluate the inverse of the beta distribution function.

L5a2c... Cauchy, chi-squared

`CHIIN` Evaluate the inverse of the chi-squared distribution function.

L5a2f ... *F* distribution

`FIN` Evaluate the inverse of the *F* distribution function.

L5a2n... Negative binomial, normal, normal scores

`ANORIN` Evaluate the inverse of the standard normal (Gaussian) distribution function.

L5a2t.... *t* distribution

`TIN` Evaluate the inverse of the Student's *t* distribution function.

L5b ...... Multivariate

L5b1 .... Cumulative distribution functions, probability density functions

L5b1n... Normal

`BNRDF` Evaluate the bivariate normal distribution function.

N.......... DATA HANDLING

N1........ Input, output

`IFNAN`.. Check if a value is NaN (not a number).

N4........ Storage management (e.g., stacks, heaps, trees)

`IWKCIN` Initialize bookkeeping locations describing the character workspace stack.

`IWKIN` Initialize bookkeeping locations describing the workspace stack.

R.......... SERVICE ROUTINES

R1 .... Machine-dependent constants

`AMACH`	Retrieve single-precision machine constants.
`DMACH`	Retrieve double precision machine constants.
`IFNAN`	Check if a value is NaN (not a number).
`IMACH`	Retrieve integer machine constants.
`UMACH`	Set or retrieve input or output device unit numbers.

R3 .... Error handling

`ERSET`	Set error handler default print and stop actions.
`IERCD`	Retrieve the code for an informational error.

Appendix B: Alphabetical Summary of Routines

IMSL MATH/LIBRARY Special Functions

ACOSH	23	Evaluate the arc hyperbolic cosine.
AI	133	Evaluate the Airy function.
AID	135	Evaluate the derivative of the Airy function.
AIDE	139	Evaluate the exponentially scaled derivative of the Airy function.
AIE	137	Evaluate the exponentially scaled Airy function.
AKEI0	124	Evaluate the Kelvin function of the second kind, kei, of order zero.
AKEI1	130	Evaluate the Kelvin function of the second kind, kei, of order one.
AKEIP0	127	Evaluate the Kelvin function of the second kind, kei, of order zero.
AKER0	123	Evaluate the Kelvin function of the second kind, ker, of order zero.
AKER1	130	Evaluate the Kelvin function of the second kind, ker, of order one.
AKERP0	126	Evaluate the derivative of the Kelvin function of the second kind, ker, of order zero.
AKS1DF	181	Evaluate the distribution function of the one-sided Kolmogorov-Smirnov goodness of fit D^+ or D^- test statistic based on continuous data for one sample.
AKS2DF	184	Evaluate the distribution function of the Kolmogorov-Smirnov goodness of fit D test statistic based on continuous data for two samples.
ALBETA	64	Evaluate the natural logarithm of the complete beta function for positive arguments.

ALGAMS	52	Return the logarithm of the absolute value of the gamma function and the sign of gamma.
ALI	31	Evaluate the logarithmic integral.
ALNGAM	49	Evaluate the logarithm of the absolute value of the gamma function.
ALNREL	6	Evaluate the natural logarithm of one plus the argument.
AMACH	240	Retrieve single-precision machine constants.
ANORDF	186	Evaluate the standard normal (Gaussian) distribution function.
ANORIN	188	Evaluate the inverse of the standard normal (Gaussian) distribution function.
ASINH	21	Evaluate the arc hyperbolic sine.
ATANH	24	Evaluate the arc hyperbolic tangent.
BEI0	122	Evaluate the Kelvin function of the first kind, bei, of order zero.
BEI1	129	Evaluate the Kelvin function of the first kind, bei, of order one.
BEIP0	125	Evaluate the derivative of the Kelvin function of the first kind, bei, of order zero.
BER0	121	Evaluate the Kelvin function of the first kind, ber, of order zero.
BER1	128	Evaluate the Kelvin function of the first kind, ber, of order one.
BERP0	124	Evaluate the derivative of the Kelvin function of the first kind, ber, of order zero.
BETA	62	Evaluate the complete beta function.
BETAI	66	Evaluate the incomplete beta function ratio.
BETDF	189	Evaluate the beta probability distribution function.
BETIN	191	Evaluate the inverse of the beta distribution function.
BI	134	Evaluate the Airy function of the second kind.
BID	136	Evaluate the derivative of the Airy function of the second kind.
BIDE	140	Evaluate the exponentially scaled derivative of the Airy function of the second kind.

`BIE`	138	Evaluate the exponentially scaled Airy function of the second kind.
`BINDF`	172	Evaluate the binomial distribution function.
`BINOM`	43	Evaluate the binomial coefficient.
`BINPR`	173	Evaluate the binomial probability function.
`BNRDF`	192	Evaluate the bivariate normal distribution function.
`BSI0`	89	Evaluate the modified Bessel function of the first kind of order zero.
`BSI0E`	95	Evaluate the exponentially scaled modified Bessel function of the first kind of order zero.
`BSI1`	91	Evaluate the modified Bessel function of the first kind of order one.
`BSI1E`	95	Evaluate the exponentially scaled modified Bessel function of the first kind of order one.
`BSIES`	107	Evaluate a sequence of exponentially scaled modified Bessel functions of the first kind with nonnegative real order and real positive arguments.
`BSINS`	100	Evaluate a sequence of modified Bessel functions of the first kind with integer order and real arguments.
`BSIS`	106	Evaluate a sequence of modified Bessel functions of the first kind with real order and real positive arguments.
`BSJ0`	84	Evaluate the Bessel function of the first kind of order zero.
`BSJ1`	86	Evaluate the Bessel function of the first kind of order one.
`BSJNS`	98	Evaluate a sequence of Bessel functions of the first kind with integer order and real arguments.
`BSJS`	103	Evaluate a sequence of Bessel functions of the first kind with real order and real positive arguments.
`BSK0`	92	Evaluate the modified Bessel function of the third kind of order zero.
`BSK0E`	96	Evaluate the exponentially scaled modified Bessel function of the third kind of order zero.
`BSK1`	93	Evaluate the modified Bessel function of the third kind of order one
`BSK1E`	97	Evaluate the exponentially scaled modified Bessel function of the third kind of order one.

BSKES	110	Evaluate a sequence of exponentially scaled modified Bessel functions of the third kind of fractional order.
BSKS	109	Evaluate a sequence of modified Bessel functions of the third kind of fractional order.
BSY0	87	Evaluate the Bessel function of the second kind of order zero.
BSY1	88	Evaluate the Bessel function of the second kind of order one.
BSYS	105	Evaluate a sequence of Bessel functions of the second kind with real nonnegative order and real positive arguments.
CACOS	16	Evaluate the complex arc cosine.
CACOSH	24	Evaluate the complex arc hyperbolic cosine.
CARG	1	Evaluate the argument of a complex number.
CASIN	15	Evaluate the complex arc sine.
CASINH	22	Evaluate the complex arc hyperbolic sine.
CATAN	17	Evaluate the complex arc tangent.
CATAN2	18	Evaluate the complex arc tangent of a ratio.
CATANH	25	Evaluate the complex arc hyperbolic tangent.
CBETA	63	Evaluate the complex complete beta function.
CBINS	102	Evaluate a sequence of modified Bessel functions of the first kind with integer order and complex arguments.
CBIS	115	Evaluate a sequence of modified Bessel functions of the first kind with real order and complex arguments.
CBJNS	99	Evaluate a sequence of Bessel functions of the first kind with integer order and complex arguments.
CBJS	112	Evaluate a sequence of Bessel functions of the first kind with real order and complex arguments.
CBKS	117	Evaluate a sequence of modified Bessel functions of the third kind with real order and complex arguments.
CBRT	2	Evaluate the cube root
CBYS	113	Evaluate a sequence of Bessel functions of the second kind with real order and complex arguments.
CCBRT	3	Evaluate the complex cube root.
CCOSH	20	Evaluate the complex hyperbolic cosine.
CCOT	12	Evaluate the complex cotangent.

CEJCN	162	Evaluate the complex Jacobi elliptic integral cn(z, m).
CEJDN	164	Evaluate the complex Jacobi elliptic integral dn(z, m).
CEJSN	159	Evaluate the complex Jacobi elliptic function sn(z, m).
CERFE	75	Evaluate the complex scaled complemented error function.
CEXPRL	5	Evaluate the complex exponential function factored from first order.
CGAMMA	46	Evaluate the complex gamma function.
CGAMR	48	Evaluate the reciprocal complex gamma function.
CHI	37	Evaluate the hyperbolic cosine integral.
CHIDF	194	Evaluate the chi-squared distribution function.
CHIIN	196	Evaluate the inverse of the chi-squared distribution function.
CI	34	Evaluate the cosine integral.
CIN	35	Evaluate a function closely related to the cosine integral.
CINH	38	Evaluate a function closely related to the hyperbolic cosine integral.
CLBETA	65	Evaluate the complex logarithm of the complete beta function.
CLNGAM	51	Evaluate the complex natural logarithm of the gamma function.
CLNREL	7	Evaluate the principal value of the complex natural logarithm of one plus the argument.
CLOG10	6	Evaluate the principal value of the complex common logarithm.
COSDG	14	Evaluate the cosine for the argument in degrees.
COT	11	Evaluate the cotangent.
CPSI	58	Evaluate the logarithmic derivative of the gamma function for a complex argument.
CSEVL	231	Evaluate the N-term Chebyshev series.
CSINH	19	Evaluate the complex hyperbolic sine.
CSNDF	197	Evaluate the noncentral chi-squared distribution function.
CTAN	10	Evaluate the complex tangent.
CTANH	20	Evaluate the complex hyperbolic tangent.

CWPL	154	Evaluate the Weierstrass *P*-function in the lemniscat case for complex argument with unit period parallelogram.
CWPLD	155	Evaluate the first derivative of the Weierstrass *P*-function in the lemniscatic case for complex argum with unit period parallelogram.
CWPQ	156	Evaluate the Weierstrass *P*-function in the equianharmonic case for complex argument with unit period parallelogram.
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ELRC	151	Evaluate an elementary integral from which inverse circular functions, logarithms and inverse hyperbolic functions can be computed.
ELRD	149	Evaluate Carlson's incomplete elliptic integral of the second kind RD(X, Y, Z).
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4. Include the name of the routine for which assistance is needed and a description of the problem

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